The Condorcet-Duverger Trade-Off: swing voters and voting equilibria.

Laurent Bouton and Micael Castanheira
ECARES, ULB

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1 Introduction

There is a striking discrepancy between the way elections operate and the way we model them. During elections, candidates campaign to advertise their ideas and platforms, and voters use this information to learn which candidate would best represent them. The information being difficult to process, many "swing" voters remain undecided or uncertain.[1] A substantial part of the divisions among the voters also stem from the different interpretations and beliefs about what are just policies.[2]

The models studying the properties of electoral systems overlook this learning process and the presence of "swing" voters. Typically, the modeler's first assumption is to endow voters with a preference ordering over candidates. This ordering is fixed by assumption: there is no learning; preferences do not change. We show that this assumption is far from innocuous: it hides the presence of some equilibria and affects the properties of others.

We focus our attention on plurality (aka *first-past-the-post*) elections and model voter preferences like in the Condorcet Jury Theorem literature (Austen-Smith and Banks 1996, Feddersen and Pesendorfer 1996, 1997, 1998, 1999, Myerson 1998a). This approach to voter preferences reflects a world in which voters may have common goals, such as improve their

[1] Probabilistic voting models implicitly recognize this by letting a part of the voting behavior be determined by a random component; see e.g. Dixit and Londregan (1998). Persson and Tabellini (2000) provide an enlightening review of these models.

[2] The dynamics of beliefs may then generate substantially different economic equilibria. See Piketty (1995), Alesina and Angeletos (2005) and Benabou and Tirole (2006) among others.

economic condition, their personal security, etc. but disagree about the best way to achieve this: voters agree about the ends but not necessarily about the means. Depending on their information, they form beliefs about which candidate would be best, but this information is imperfect. The difference with the standard approach is thus that the relationship between voter preferences and their ranking of candidates is fuzzier; it depends on an unobservable state of nature that determines which candidate is "truly" best for the swing voters.

We consider three types of voters who support three different candidates. Types t_A and t_B support candidates A and B respectively, and rank candidate C as their worst candidate. The division among t_A and t_B voters is information-based: they hold opposite beliefs about the true state of nature, but have identical underlying preferences. In other words, they would agree on which candidate is truly the best if they held identical beliefs. Those will be the *swing voters*: if the election revealed sufficient information, some would change their mind. Types t_C are *stalwart voters*. They always support C. Those are a minority: the election of C is thus the worst possible outcome.

In section 3, we begin by reviewing two-candidate elections to briefly summarize the literature on the Condorcet Jury Theorem. In section 4, we study three-candidate elections. We highlight an important trade-off between "vote division", which is a necessary condition for the Condorcet Jury Theorem to be valid, and "Duvergerian" forces that induce swing voters to coordinate all their votes on a single candidate, to beat C. This is what we call the *Condorcet-Duverger trade-off.* Under plurality, the "Duvergerian" forces may be either too strong or too weak. If they are too strong, the election of the best candidate can be jeopardized, because majority voters do not sufficiently divide their voters. If they are too weak, swing voters may coordinate insufficiently and let C win the election.

This Condorcet-Duverger trade-off in plurality elections shows why it is important to take account of swing voters in the analysis of any electoral system. In Bouton and Castanheira (2008), we study other systems: in the spirit of Myerson and Weber (1993), we compare the equilibrium properties of plurality elections with that of run-off elections and of Approval Voting. We show that the Condorcet-Duverger trade-off is still present under run-off but is absent under Approval Voting. Under the latter system, there is a unique voting equilibrium, and therefore no possible coordination failure, and the Condorcet Jury Theorem holds. There is thus *full information and coordination equivalence.*

2 The model

We study the equilibrium properties of plurality elections when there are swing voters and compare these properties with that of a model in which the presence of swing voters is overlooked. In plurality elections, the candidate receiving the most votes is elected. Ties are resolved by the toss of a fair coin.

We conduct our analysis under the assumption that the total number of voters is distributed according to a Poisson distribution with some mean n (see Myerson 1998b and 2000 for the properties of Poisson Games). The probability that there are k voters in the population is therefore:

$$\Pr(k|n) = \frac{e^{-n}\, n^k}{k!}.$$

There are three types of voters $t \in \{t_A, t_B, t_C\}$ and two states of nature: $\omega \in \{a, b\}$. We are thus analyzing an *extended Poisson Game* as introduced by Myerson (1998a), in which the probability that a given voter has type t depends on the actual state of nature ω. These probabilities are denoted $r(t|\omega)$, with $\sum_t r(t|\omega) = 1$. The actual state of nature is unknown to the voters. They only know that the probability of state ω is $q(\omega) \in [0,1]$, s.t. $q(a) + q(b) = 1$.

There will be up to three candidates, $P = A$, B and C. We denote the utility of the voters by the function $U(P, t, \omega)$, where P is the party winning the election, t is the voter's type, and ω is the state of nature. An *equilibrium* is found when each voter's strategy maximizes her expected utility given the vote share of each party, and these vote shares are coherent with the voters' actions (Myerson and Weber 1993, Myerson 2002). Note that the act of voting is costless; if it happens in equilibrium, abstention thus reveals that some votes would strictly reduce the voters' expected utility.

2.1 Voter types: minority and majority blocks

Types t_C represent the *minority block*. They are *stalwart* in the sense that they prefer candidate C independently of the state of nature. For the sake of tractability, we also assume that they are indifferent between the other two candidates:

$$\begin{aligned} U(P, t_C, \omega) &= 1 \text{ if } P = C \\ &= 0 \text{ if } P \in \{A, B\}. \end{aligned}$$

Since they are a minority, the probability that a given voter has type t_C must be below one half: we impose $r(t_C|a) = r(t_C|b) < 1/2$.

Types t_A and t_B together represent the *majority block.* They are *swing voters* in the sense that their preferences over candidates, as well as their share in the electorate, are state-contingent. There are more types t_A in state a than in state b: $r(t_A|a) > r(t_A|b)$, and conversely for types t_B. By Bayesian updating, a type t voter thus infers that:

$$q(\omega|t) = \frac{q(\omega)\, r(t|\omega)}{q(a)\, r(t|a) + q(b)\, r(t|b)}.$$

Since $r(t_A|a) > r(t_A|b)$ and $r(t_B|a) < r(t_B|b)$, types t_A and t_B hold different beliefs about the likelihood of the two states of nature. To introduce divisions within the majority, we impose that types t_A and t_B support a different candidate:

$$\frac{q(a|t_A)}{q(b|t_A)} > 1 > \frac{q(a|t_B)}{q(b|t_B)}. \tag{1}$$

Conditional on the state of nature, their preferences are aligned: for types $t \in \{t_A, t_B\}$, we have

$$\begin{aligned} U(P,t,\omega) &= 1 \text{ if } (P,\omega) = (A,a) \text{ or } (B,b) \\ &= 0 \text{ if } (P,\omega) = (A,b) \text{ or } (B,a) \\ &= -1 \text{ if } P = C. \end{aligned}$$

Thus, majority-block voters have common views about the objectives that policy should pursue but they have opposite priors about the means to reach these objectives; they have opposite views regarding the true state of nature. In other words, the mapping between objectives and candidates is blurred; it depends on an unobservable state of nature. This is why we call these voters "swing". With sufficiently convincing information, they may admit that their priors were wrong, and modify their support for one or the other candidate.

In contrast, type t_C voters pursue different policy objectives. This is why both majority-block types always agree that the election of alternative C would be the worst possible outcome.

Finally, we make the technical assumption that, on average, there can be more types t_A than types t_B in the population:

$$r(t_A|a) + r(t_A|b) \geq r(t_B|a) + r(t_B|b).$$

This assumption is only meant to ensure that our results do not hinge on any type of symmetry across types.

2.2 Strategy set and action profiles

The voters' action set is denoted $\Psi = \{\varnothing, A, B, C\}$. That is, voters can either abstain or vote for one of the three candidates. Let $\sigma(\psi|t)$ denote the probability that a player plays $\psi \in \Psi$ if he has type t. The usual constraints apply: $\sigma(\psi|t) \geq 0$ and $\sum_{\psi} \sigma(\psi|t) = 1$, $\forall t$. For short $\sigma(t)$ will denote the vector of these probabilities.

Aggregating strategies, the *expected share* of voters playing action ψ in state ω is therefore:

$$\tau(\psi|\omega) = \sum_t r(t|\omega)\ \sigma(\psi|t).$$

Remark that these expected fractions can differ across states, but only because the fraction of each type $r(t|\omega)$ is state-dependent.

Thus, there is an *informational trap* if t_A and t_B voters adopt the same strategy $\sigma(t)$. In that case, the expected result of the election is the same in both states of nature; observing the election outcome cannot reveal additional information about the actual state of nature. Under an informational trap, voters with different types cannot eventually agree on a candidate. When there is no informational trap, the outcome of the election reveals a lot of information about the actual state of nature. In that case, swing voters expect the election to potentially modify their priors.

Following Myerson (1998b, 2000), if the expected size of the population is n and if *expected shares* are $\tau(\psi|\omega)$, then the *realized number of votes* for ψ, $x(\psi)$, is a Poisson variable with mean $n\,\tau(\psi|\omega)$:

$$\Pr(x(\psi) \mid \tau(\cdot)) = \frac{e^{-n\,\tau(\psi|\omega)}\ [n\,\tau(\psi|\omega)]^{x(\psi)}}{x(\psi)!}.$$

This distribution depends on the voters' strategy σ and on the state of nature ω.

Using Theorem 1 of Myerson (2000) we can characterize the limiting probability that the number of votes for each action is some vector $\vec{x} = (x(A), x(B), x(C), x(\varnothing))$. This probability centrally depends on the *magnitude,* denoted *mag*, of the considered event $\vec{x}$:

Property 1 *(Myerson 2000, Theorem 1) Subject to* $\sum_{\psi \in \{\varnothing, A, B, C\}} \tau(\psi|\omega) = 1$, $\omega \in \{a, b\}$, *and given expected shares* $\tau(\omega)$, *the probability that the actual number of votes is* $\vec{x} =$

$(x(A), x(B), x(C), x(\varnothing))$ *converges to*:

$$\Pr(\vec{x}|\tau(\omega)) \underset{n\longrightarrow\infty}{\longrightarrow} \max_{\vec{x}} \frac{\exp[mag\,[\vec{x}]\,n]}{\prod_{\psi}\sqrt{2\pi x(\psi)+\frac{\pi}{3}}},$$

$$\textit{where: } mag\,[\vec{x}] = \sum_{\psi}\frac{x(\psi)}{n}\left(1-\log(\frac{x(\psi)}{n\ \tau(\psi|\omega)})\right)-1 \quad (\leq 0)$$

Our results only exploit this magnitude theorem and are thus valid for basically any distribution of voters that generate the same comparative statics. The comparative statics implication of the magnitude theorem is that the probability of an event is exponentially decreasing in population size n. Therefore, if we compare two events, call them 1 and 2, with magnitudes $mag_1 > mag_2$, then event 1 will become infinitely more likely to happen than event 2.

Formally, $\lim_{n\to\infty} \log[\Pr(\vec{x}|\tau(\omega))]/n = mag\,[x]$. Hence, the most likely events are those with magnitude 0. The event with magnitude 0 is $\vec{x} = \tau(\omega)\cdot n$, i.e. the event that actual vote shares are arbitrarily close to expected vote shares. The magnitude of all the other events is strictly negative.

3 Two-candidate elections

This section reviews some of the main results in the Condorcet Jury Theorem literature. This literature primarily focused on two-candidate elections and showed how the presence of swing voters affects the equilibrium properties of the election. The fundamental change is that voters may prefer to vote against their *a priori* preferred alternative to enhance election efficiency. That is, two-candidate elections generate *full information equivalence*: the winning alternative is the one that would have been chosen under full information (Austen-Smith and Banks 1996, Feddersen and Pesendorfer 1997 and Myerson 1998a). Full information equivalence requires that A ranks first in state a and B ranks first in state b.

To summarize these findings, we can shrink the fraction of t_C-voters to zero in our model: this produces a two-candidate setup that is almost identical to the one of Myerson (1998a). The difference between our setup and Myerson's is that we allow for abstention. We show that voters never vote against their *a priori* preferred alternative, because abstention dominates such cross-voting. In the next section, we extend the analysis to a three-candidate setup.

The *expected* value of a ballot is $G(\psi|t)$. It depend on the probability of being pivotal against the other candidate. The value of a vote for candidate A is:

$$G(A|t) = q(a|t)\Pr(piv_{AB}|a) - q(b|t)\Pr(piv_{AB}|b), \tag{2}$$

which reads as follows: a type t expects that the state of nature is a with probability $q(a|t)$. In that state, a majority block voter's utility is 1 if A wins and 0 if B wins. Therefore, if the vote is pivotal (in favor of A, against B), utility **increases** by 1. If the actual state is b, utility **decreases** by 1. The value of a vote for B is derived in the same way:

$$G(B|t) = q(b|t)\Pr(piv_{BA}|b) - q(a|t)\Pr(piv_{BA}|a). \tag{3}$$

Note the difference between a swing voter and a stalwarts voter: the swing voter is trying to elect the best candidate, while the stalwarts voter is only trying to elect her candidate. Stalwarts voters can be represented as voters who assign probability 0 to the "other" state of nature. That is, stalwarts types t_A would assign a probability 1 to a and a probability 0 to b, and the opposite for stalwarts types t_B. These stalwarts types have a simple dominant strategy: vote for their own candidates.

Austen-Smith and Banks (1996) showed that this strategy is generally not an equilibrium for swing voters. In a setup where abstention is not allowed, types t_A develop an incentive to vote with strictly positive probability for the candidate they like least. The idea is that, since t_A's are more abundant, the outcome of the election would be biased in favor of candidate A if everyone voted "sincerely". Swing voters prefer to compensate this bias, in order to maximize the probability that the best candidate is elected.

In our setup, we can show that such a strategy is also dominated by a strategy of mixing between abstention and voting for one's own candidate. The following lemma establishes that:

Lemma 1 *In equilibrium, voters never mix between A and B.*

Proof. See Appendix. ■

This result is a manifestation of the swing voter's curse identified by Feddersen and Pesendorfer (1996, 1999). The difference between their result and ours is that we do not consider voters with different information qualities. Combining their result and ours shows that, if different voters had different information qualities, those with worse information would abstain more.

Proposition 1 shows that the equilibrium of the voting game with swing voters is unique:[3]

Proposition 1 *The equilibrium of the 2-candidate election game is unique and such that types-t_A may only mix between $\psi = A$ and abstention, while types-t_B play $\psi = B$ in pure strategy. This equilibrium features full information equivalence.*

Proof. See Appendix. ■

The literature focused on a population only composed of swing voters.[4] Yet, it would be straightforward to incorporate stalwarts voters who only vote A or B: swing voters would simply compensate the votes of stalwarts types by "leaning against the wind". That is, if there were more stalwarts t_A, then swing t_A's would have to abstain more often. The only constraint would be that the fraction of swing voters is sufficiently large for full information equivalence to be sustained.

Two corollaries result from this proposition. First, Austen-Smith and Banks (1996) have shown that the only case in which voters vote sincerely is when $r(t_A|a) = r(t_B|b)$. The same result holds in our setup: abstention is also the result of an imbalance between the proportion of the two types across states of nature:

Corollary 1 *In equilibrium, types-t_A abstain with positive probability if and only if $r(t_A|a) > r(t_B|b)$.*

Second, we have seen that full information equivalence requires that A be first is state a and B be first in state b. Hence:

Corollary 2 *A necessary condition for full information equivalence is that swing voters split their votes between A and B.*

[3] Feddersen and Pesendorfer (1999) have a more general model, in which they provide sufficient conditions for full information equivalence to hold in equilibrium (Proposition 4).

[4] An exception is Castanheira (2003), in which voters have fixed preferences. Yet, voters have a incentive to vote for losers, in order to inform parties that they are sufficiently numerous. This may attract parties closer to their preferred policy in subsequent elections.

4 Three-candidate elections

The results of the previous section show that if there are sufficiently many swing voters, elections will select the candidate who is socially preferred. That is, the winning candidate is the same as if there were no information imperfections. This full information equivalence was shown to extend to qualified majorities, as long as unanimity is not required (Feddersen and Pesendorfer 1997, 1998, 1999).

Yet, this literature largely overlooked multicandidate elections (two exceptions are Piketty 2000 and Castanheira 2003). In this section, we show how the properties of two-candidate elections are altered when a third candidate enters the electoral race. With a third candidate, our setup is similar to the one-period case analyzed by Piketty (2000). The difference, again, is that we introduce abstention. Also, we shed new light on the properties of the various equilibria that emerge and prove the existence of new equilibria.

We raise 3 issues. The first one is the best know strategic effect of plurality elections: the existence of a third candidate is sufficient to generate "Duvergerian" equilibria, in which either A or B receive zero vote. In these equilibria, there is of course no full information equivalence (see also Piketty 2000, Proposition 5). Second, we investigate the properties of a "Condorcet-Jury" type of equilibrium, in which the three candidates receive a strictly positive vote share. We show that this equilibrium is stable and produces full information equivalence if the vote share of C, is sufficiently low. If the vote share of C is too large, information aggregation is impossible, and this equilibrium becomes unstable. Third, we prove the existence of additional equilibria with *partial information equivalence*, when the vote share of C is not too large.

4.1 Issue 1. Duvergerian equilibria always exist

The equilibrium properties of plurality elections have been widely analyzed in a setup without swing voters (see for instance Riker 1982, Myerson and Weber 1993). One of the main results of that analysis is the validation of *Duverger's Law* (Duverger 1954): strategic motivations induce some voters to abandon their preferred candidate and to focus their votes on the top-two candidates, with the largest vote shares. In our setup, this implies that there is one equilibrium in which all majority types vote for A, and another equilibrium in which they all vote for B. These two equilibria are stable. A third, knife-edge, equilibrium is that majority type voters divide their votes equally between A and B.

Proposition 2, which is reminiscent of Piketty (2000,¨Proposition 5), shows that such equilibria also exist in a setup with swing voters:

Proposition 2 *In multicandidate elections, Duvergerian equilibria always exist. That is,* $\tau(A|\cdot) = 0$ *and* $\tau(B|\cdot) = 0$ *are (self-fulfilling) equilibria. These equilibria are inefficient, because they prevent learning and the election of the candidate who would be chosen under full information.*

To prove this result, first note that majority-type voters play A (respectively: B) with probability 1 if the following pay-off difference is strictly positive (resp. negative):

$$\begin{aligned} G(A|t) - G(B|t) = \quad & q(a|t) \ \{2\Pr(piv_{AC}|a) - \Pr(piv_{BC}|a) \\ & \qquad + \Pr(piv_{AB}|a) + \Pr(piv_{BA}|a)\} \\ & + q(b|t) \ \{\Pr(piv_{AC}|b) - 2\Pr(piv_{BC}|b) \\ & \qquad - \Pr(piv_{AB}|b) - \Pr(piv_{BA}|b)\}. \end{aligned} \tag{4}$$

To show the existence of Duvergerian equilibria, we thus have to show that the difference is strictly positive when $\tau(B|\omega)$ is sufficiently small and conversely for $\tau(A|\omega)$ small. To this end, we show that the magnitude of the events $piv_{BC}|\omega$ and $piv_{BA}|\omega$ are smaller than the magnitude of $piv_{AC}|\omega$ when $\tau(B|\omega)$ is sufficiently small.

Using the properties of Poisson Games, we actually find that the probability of being pivotal in favor of the candidate with the smallest vote share actually becomes *infinitesimally* smaller than the one in favor of the leading candidate:

Lemma 2 *The magnitude of the pivot probability between two parties P and Q is:*

$$mag(piv_{PQ}|\omega) = -\left(\sqrt{\tau(P|\omega)} - \sqrt{\tau(Q|\omega)}\right)^2,$$

if these two parties are the top-two candidates, and it is smaller than that value for the bottom-two candidates.
Hence, if the three parties have different vote shares: $\tau(P|\omega) > \tau(Q|\omega) > \tau(R|\omega)$, the following pivot probability ratio converges to infinity as population size increases:

$$\lim_{n\to\infty} \Pr \frac{(piv_{PQ}|\omega)}{\max\{\Pr(piv_{PR}|\omega), \Pr(piv_{QR}|\omega)\}} = \infty.$$

Proof. See appendix. ■

This lemma is then largely sufficient to establish that (4) is necessarily positive if the vote share of B is sufficiently small, and negative if the vote share of B is sufficiently large.

In other words, a given voter who best responds to the voting patterns in the rest of the electorate should follow the lead of the majority, and abandon trailing candidates. This is why the two Duvergerian equilibria are self-fulfilling: it is the *expectation* that a candidate will be trailing behind that triggers this response.[5]

4.2 Issue 2. Full information equivalence can be impossible to attain

When there are only two candidates, full information equivalence only requires that the equilibrium vote share of candidate A be larger than that of candidate B in state a and conversely in state b. Matters become more complex when a strictly positive fraction of types t_C vote for the third candidate, C. In that case, full information equivalence requires that:

$$\begin{cases} \tau(A|a) > \max\left[\tau(B|a), \tau\left(C\right)\right], \\ \tau(B|b) > \max\left[\tau(A|b), \tau\left(C\right)\right]. \end{cases} \tag{5}$$

That is, the leader's vote share –A's or B's, depending on the state– must also be above C's. This constraint is far from being trivial: as we show below, full information equivalence is actually impossible to reach when $\tau\left(C\right)$ is large. By contrast, when $\tau\left(C\right)$ is small, full information equivalence is not only feasible; it is actually a stable equilibrium.

4.2.1 Case 1. $\tau\left(C\right)$ is large

If the fraction of types t_C is large enough, C will necessarily be one of the top-two candidates. By virtue of Lemma 2, this implies that the pivotability between A and B, which was central to the results for two-candidate elections, is not a second-order concern for the voters. The largest magnitude is necessarily associated to one of the pivotabilities against C.

In this case, the only non-Duvergerian equilibrium of the game is unstable and requires that A and B have the same vote share in "their" respective state:[6]

[5] Remark that abstention is also a dominated action in these equilibria.

[6] "Stability" here is used in the same way as in a Cournot equilibrium: assume that expected vote shares are $\tau^0(P|\omega)$. Given $\tau^0(P|\omega)$, allow a tiny fraction of the electorate to choose their strategy, and then compute the new expected vote shares $\tau^1(P|\omega)$, and iterate to identify a sequence $\tau^k(P|\omega)$, $k = 1, 2, ...$

Proposition 3 *For* $\tau(C) > 1/[2 + r(t_A|b)/r(t_A|a)]$*, abstention is a dominated strategy and the only non-Duvergerian equilibrium is such that:*

$$\tau(C) > \tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0.$$

This equilibrium is not "stable" and does not produce full information equivalence, since it induces the election of the dominated candidate C *with a probability that converges to 1 as* $n \to \infty$.

Proof. See appendix. ■

[FIGURE 1 ABOUT HERE]

Figure 1 illustrates this result: the horizontal axis displays the strategy of types t_A (as seen in the previous section, types t_B vote for B with probability 1). Moving from left to right, they vote for B with increasing probability. The upward sloping lines represent B's vote share in each state of nature. The downward sloping lines represent A's vote share.

The equilibrium, represented by point E on the graph, is such that $\tau(A|a) \simeq \tau(B|b)$. To the left of that point, the top two contenders are A and C, and the pivot probability $\Pr(piv_{AC}|a)$ is –by orders of magnitude– larger than the other pivot probabilities. Hence, majority-group voters would strictly prefer to deviate by voting for A only. Conversely, the same holds for B to the right of E. By a fixed point argument, they must be indifferent between A and B at point E, which proves the existence of the equilibrium. Note that the threshold $1/[2 + r(t_A|b)/r(t_A|a)]$ simply identifies the vote share $\tau(C)$ such that $\tau(C) = \tau(A|a) = \tau(B|b)$ at point E.

4.2.2 Case 2. $\tau(C)$ is small

When $\tau(C)$ is below that threshold, there exists a range of strategies around point E such that both $\tau(A|a)$ and $\tau(B|b)$ are larger than $\tau(C)$. In that case, the equilibrium at point E is actually stable:

We call an equilibrium "stable" if there exists a neighbourhood of the equilibrium $\tau^*(P|\omega)$ such that the sequence $\tau^k(P|\omega)$ converges to $\tau^*(P|\omega)$.

Proposition 4 *For* $\tau(C) < 1/[2 + r(t_A|b)/r(t_A|a)]$*, there exists a "stable" equilibrium with full information equivalence: vote shares are such that*

$$\tau(A|a) \simeq \tau(B|b) > \max[\tau(C), \tau(A|b) \simeq \tau(B|a)] > 0.$$

Sketch of the proof:

The proof itself is quite straightforward, although tedious. So, we only develop a graphic argument that illustrates how the proof proceeds.

[FIGURE 2 ABOUT HERE]

In Figure 2, $\tau(C)$ has a value below $1/[2+r(t_A|b)/r(t_A|a)]$. The important implication of this smaller value is that the ranking of pivot probabilities is now opposite to the ranking we identified in Figure 1. Indeed, at point D in Figure 2, we have $\tau(A|a) > \tau(B|b) > \tau(C)$. This means that A beats C by a large margin in state a but B only beats C by a small margin in state b. Hence, the pivot probability in favor of B in state b dominates, by orders of magnitude. This implies that $G(A|t) - G(B|t)$ in (4) is necessarily negative: if the vote share of B falls slightly below its equilibrium level, all voters wish to vote for B. In other words, a decrease in $\sigma(B|t_A)$ induces a Nash response that increases this value back to point E: the equilibrium is "stable" in that sense. A similar mechanism holds to the right of E. ■

Interestingly, if $\tau(C)$ is also smaller than $1/[2 + r(t_A|a)/r(t_A|b)]$, then C would have the smallest vote share of the three alternatives. This implies that, at point E on the figures, the voters will behave "as if" C was not present in the electoral race. Concretely, this implies that t_A voters must be abstaining with some probability at the equilibrium.

Remark 1 *If abstention is not in the action set and if the fraction of types* t_C *is sufficiently small,* t_A*-voters use* C *as a surrogate for abstention.* t_A*-voters would then vote* C *with positive probability.*

4.3 Issue 3. Existence of additional equilibria

Propositions 3 and 4 emphasize little-known or unknown properties of the "Condorcet equilibrium". Here, we prove the existence of additional equilibria when $\tau(C)$ is small. In these equilibria, majority-type voters divide their votes between A and B in such a way that C wins in one of the two states of nature:

Proposition 5 *For $1/3 < \tau(C) < 1/[2 + r(t_A|b)/r(t_A|a)]$, there exist two "unstable" equilibria with* ***partial*** *information equivalence: vote shares are such that*

$$\tau(B|b) > \tau(C) > \tau(A|a) = \tau(B|a) > 0 \text{ in one equilibrium, and}$$
$$\tau(A|a) > \tau(C) > \tau(A|b) = \tau(B|b) > 0 \text{ in other equilibrium.}$$

In the former equilibrium, B wins in state b and C wins in state a. In the latter, A wins in state a and C wins in state b.

The proof is again both straightforward and tedious, which is why we only develop a sketch of the proof.

Sketch of the proof:

Figure 3 relies on the same parameter values as Figure 2. The only difference is thus that it points towards another equilibrium, represented by point F on the Figure.

[FIGURE 3 ABOUT HERE]

For a strategy $\sigma(B|t_A)$ slightly to the left of F, we have: $\tau(C) > \tau(A|a) > \tau(B|a)$ in state a, and $\tau(B|b) > \tau(C) > \tau(A|b)$ in state b. The two pivot probabilities that dominate are therefore $\Pr(piv_{AC}|a)$ and $\Pr(piv_{BC}|b)$. Yet, the gap between A and C in state a is smaller than the gap between B and C in state b. Therefore: $\Pr(piv_{AC}|a) >>> \Pr(piv_{BC}|b)$, which implies that a vote for A is more valuable than a vote for B.

Conversely, for a strategy $\sigma(B|t_A)$ slightly to the right of F, we have: $\tau(C) > \tau(B|a) > \tau(A|a)$ in state a (nothing changes in state b), which implies that a vote for B is now more valuable than a vote for A. By a fixed point argument, it is immediate that t_A voters must be indifferent between voting A and B at point F. By symmetry, there also exists a similar point F' for $\sigma(B|t_A) = 0$ and $\sigma(B|t_B) \in (0,1)$, in which A wins in state a and C wins in state b. ■

Note also that, if $\tau(C)$ is smaller than 1/3, these equilibria still exist but generate a different type of outcome. In the equilibrium similar to point F in Figure 3, B would win in both states of nature. Thus, like in a Duvergerian equilibrium, the winning alternative is independent of the state of nature but, unlike a Duvergerian equilibrium, the three candidates receive a strictly positive vote share.

5 Conclusion

We have studied voting equilibria in plurality elections when swing voters compose a majority of the electorate. With the help of a simple model with 2 states of nature and three candidates, we have showed that new equilibria arise and that the properties of some of the existing equilibria are modified. We have also showed that swing voters tend to adopt different abstention rates depending on the equilibrium that is played.

The comparison between these equilibria also underlines that some of them are efficient, whereas other equilibria are not. Some candidates may win only due to information imperfections and coordination failures. That is, if voters were fully informed about the state of nature and could decide on which equilibrium to coordinate, they would elect one candidate in some cases, and another candidate in other cases. This is not possible under plurality elections. As underlined by Duverger (1954), voters develop an incentive to coordinate their votes on a "strong" candidate in plurality elections.

This coordination process is not only at the roots of the equilibrium multiplicity, it also prevents collective learning by the electorate. This is what we call the *Duverger-Condorcet trade-off*: ensuring the election of a candidate requires voters to coordinate their votes on that candidate. By contrast, information aggregation requires them to divide their votes across different candidates, so as to reveal the different elements of information scattered in the electorate.

To conclude, the "known" properties of electoral systems appear to actually overlook the presence of swing voters. In future work, it will thus prove important to study the properties of other electoral systems in such a setup. As explained in the introduction, we make a first step in that direction in Bouton and Castanheira (2008). We show that under Approval Voting, there is no Duverger-Condorcet trade-off: there is full information and coordination equivalence. We contend that this is a strong argument in favor of electoral reform.

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APPENDIX

5.1 Proof of Lemma 1

Proof. We show that when type-t voters prefer to abstain rather than to mix between A and B:

$$G(A|t) = G(B|t) \Longrightarrow G(B|t) < 0. \tag{6}$$

By Myerson's offset theorem:

$$\Pr(piv_{BA}|\omega) = \Pr(piv_{AB}|\omega)\sqrt{\frac{\tau(A|\omega)}{\tau(B|\omega)}}.$$

Then, from (2) and (3), we have that $G(A|t) = G(B|t)$ boils down to

$$\frac{q(a|t)}{q(b|t)} = \frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)}\frac{\sqrt{\tau(B|a)}}{\sqrt{\tau(B|b)}}.$$

Substituting for $q(a|t)$ in (2) yields:

$$G(A|t) = q(b|t)\Pr(piv_{AB}|b)\left(\frac{\sqrt{\tau(B|a)}}{\sqrt{\tau(B|b)}} - 1\right).$$

Since $\tau(B|b) > \tau(B|a)$, we have that $G(A|t) = G(B|t)$ implies $G(A|t) = G(B|t) < 0$, and hence that abstention is preferred. ■

5.2 Proof of Proposition 1

Proof. First, observe that $\sigma(B|t_A) > 0$ implies $\sigma(B|t_B) = 1$. From (2) and (3) we have that $G(B|t_A) > G(A|t_A)$ if

$$\frac{q(a|t_A)}{q(b|t_A)} < \frac{\Pr(piv_{AB}|b) + \Pr(piv_{BA}|b)}{\Pr(piv_{AB}|a) + \Pr(piv_{BA}|a)}.$$

Since $\frac{q(a|t_A)}{q(b|t_A)} > \frac{q(a|t_B)}{q(b|t_B)}$, we have that $G(B|t_A) > G(A|t_A)$ implies $G(B|t_B) > G(A|t_B)$. Similarly, we can proof that $G(B|t_A) > 0$ implies $G(B|t_B) > 0$. By analogy, we have that $\sigma(A|t_B) > 0$ implies $\sigma(A|t_B) = 1$.

Second, we show that $\sigma(B|t_A) > 0$ cannot be true in equilibrium. Since $\sigma(B|t_A) > 0$ implies $\sigma(B|t_B) = 1$, which implies that $G(A|t) > G(B|t)$ and $0, \forall t \in \{t_A, t_B\}$. The proof is similar for types t_B. This show that neither A nor B are deserted by the voters. Hence, in equilibrium we must have:

$$\begin{aligned} &G(A|t_A) \geq 0, \text{ and} \\ &G(B|t_B) \geq 0. \end{aligned} \tag{7}$$

Since $G(A|t) \gtreqless 0, \forall t \in \{t_A, t_B\}$ when $mag(piv_{AB}|A) \gtreqless mag(piv_{AB}|b)$, a necessary condition for $G(A|t) = 0$ is therefore:

$$mag(piv_{AB}|a) = mag(piv_{AB}|b). \tag{8}$$

This is satisfied when

$$-\left(\sqrt{\tau(A|a)}-\sqrt{\tau(B|a)}\right)^2=-\left(\sqrt{\tau(B|b)}-\sqrt{\tau(A|b)}\right)^2 \tag{9}$$

Since $\sigma(A|t_A)+\sigma(\varnothing|t_A)=1$ and $\sigma(B|t_B)+\sigma(\varnothing|t_B)=1$, we have that $\tau(A|a)>\tau(A|b)$ and $\tau(B|a)<\tau(B|b)$. Then, (9) is satisfied iff

$$\frac{\sigma(A|t_A)}{\sigma(B|t_B)}=\left(\frac{\sqrt{r(t_B|b)}+\sqrt{r(t_B|a)}}{\sqrt{r(t_A|a)}+\sqrt{r(t_A|b)}}\right)^2. \tag{10}$$

We still need to show that the equilibrium is unique. To do that, we have to prove that when (10) is satisfied, $G\left(B|t_B\right)>0$. By assumption, $\left(\frac{\sqrt{r(t_B|b)}+\sqrt{r(t_B|a)}}{\sqrt{r(t_A|a)}+\sqrt{r(t_A|b)}}\right)^2\leq 1$. Therefore, $\sigma(A|t_A)<1$ is necessary in equilibrium otherwise (10) would imply $\sigma(B|t_B)>1$. In equilibrium, we must then have $G\left(A|t_A\right)=0$. From (6) and (7), $G\left(A|t_A\right)=0$ directly implies that $G\left(B|t_B\right)>0$. ■

5.3 Proof of Lemma 2

Proof. First, we use Property 1 to compute the magnitude of the probability that P and Q have the same vote share in the state ω:

$$\begin{aligned} mag(piv_{PQ}|\omega) &= \max_{\vec{x}}\sum_{\psi}\frac{x\left(\psi|\omega\right)}{n}\left(1-\log\frac{x\left(\psi|\omega\right)}{n\tau\left(\psi|\omega\right)}\right)-1 \\ & \quad s.t.\ x\left(P|\omega\right)=x\left(Q|\omega\right) \end{aligned} \tag{11}$$

If we denote $x\left(P|\omega\right)=x\left(Q|\omega\right)=x(\omega)$ we find that this is maximized in: $x^{**}_{PQ}\left(\omega\right)=n\sqrt{\tau(P|\omega)\tau(Q|\omega)}$, $x^{**}\left(R|\omega\right)_{PQ}=n\tau(R|\omega)$, $x^{**}\left(\varnothing|\omega\right)_{PQ}=n\tau(\varnothing|\omega)$.

Second, a vote can only be pivotal between P and Q if the third candidate, R, has fewer votes than P and Q. This imposes an additional condition: $x\geq x\left(R|\omega\right)$. Introducing that condition in the maximization problem, we find

$$\begin{aligned} \text{If } \sqrt{\tau(P|\omega)\tau(Q|\omega)} &\geq \tau(R|\omega), \text{ then } \begin{cases} x^{*}_{PQ}\left(\omega\right)=x^{**}_{PQ}\left(\omega\right) \\ x^{*}\left(R|\omega\right)_{PQ}=x^{**}\left(R|\omega\right)_{PQ} \end{cases} \\ \text{If } \sqrt{\tau(P|\omega)\tau(Q|\omega)} &< \tau(R|\omega), \text{ then } \begin{cases} x^{*}_{PQ}\left(\omega\right)=1/3 \\ x^{*}\left(R|\omega\right)_{PQ}=1/3 \end{cases} \end{aligned}$$

where the * refers to the solution of the maximization problem that takes the new condition into account. (Note that $mag(piv_{PR}|\omega)$ and $mag(piv_{QR}|\omega)$ can be computed in the same way).

Whenever $\sqrt{\tau(P|\omega)\tau(Q|\omega)}\geq\tau(R|\omega)$, $mag(piv_{PQ}|\omega)$ is:

$$mag\left(piv_{PQ}|\omega\right)=-\left(\sqrt{\tau\left(P|\omega\right)}-\sqrt{\tau\left(Q|\omega\right)}\right)^2$$

while if $\sqrt{\tau(P|\omega)\tau(Q|\omega)}<\tau(R|\omega)$, the magnitude is:

$$mag(piv_{PQ}|\omega)=3[\tau\left(P|\omega\right)\tau\left(Q|\omega\right)\tau\left(R|\omega\right)]^{1/3}+\tau(\varnothing|\omega)-1$$

Since $\tau(P|\omega) > \tau(Q|\omega) > \tau(R|\omega)$, we have that $x^{**}_{PQ}(\omega) > 1/3$, and that $x^{**}_{QR}(\omega) < 1/3$. Therefore, $x^{*}_{PQ}(\omega) = x^{**}_{PQ}(\omega)$, and $x^{*}_{QR}(\omega) = 1/3$, which implies that:

$$\begin{aligned} mag\,(piv_{PQ}|\omega) &= -\left(\sqrt{\tau\,(P|\omega)} - \sqrt{\tau\,(Q|\omega)}\right)^2 \\ mag(piv_{QR}|\omega) &= 3[\tau\,(P|\omega)\,\tau\,(Q|\omega)\,\tau\,(R|\omega)]^{1/3} + \tau(\varnothing|\omega) - 1 \end{aligned}$$

Concerning $mag(piv_{PR}|\omega)$, the situation is a priori unclear since $x^{**}_{PR}(\omega) \in (0, 1/2)$. Nonetheless, $\left(\sqrt{\tau\,(P|\omega)} - \sqrt{\tau\,(Q|\omega)}\right)^2 < \left(\sqrt{\tau\,(P|\omega)} - \sqrt{\tau\,(R|\omega)}\right)^2$ implies that $mag(piv_{PQ}|\omega) > mag(piv_{PR}|\omega)$ when $x^{*}_{PR}(\omega) = x^{**}_{PR}(\omega)$. Since $x^{*}_{PR}(\omega) = 1/3$ results from an additional constraint, it is obvious that $mag(piv_{PR}|\omega)$ computed for this value of $x^{*}_{PR}(\omega)$ is smaller or equal to the unrestricted value, that is when computed in $x^{*}_{PR}(\omega) = x^{**}_{PR}(\omega)$. This reinforces the inequality.

Finally, since $mag(piv_{QR}|\omega)$ is identical to the restricted magnitude of $mag(piv_{PR}|\omega)$, it follows directly that $mag\,(piv_{PQ}|\omega) > mag(piv_{PR}|\omega) \geq mag(piv_{QR}|\omega)$. ■

5.4 Proof of Proposition 3

Proof. We first prove that such an equilibrium exists and necessarily entails $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$. That is, there is a *unique* non-Duvergerian equilibrium. Second, we verify whether it satisfies the "stability" properties.

Step 1. $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$ is an equilibrium

If $\tau\,(C) > 1/[2 + r\,(t_A|b)\,/r\,(t_A|a)]$, all the strategy profiles that leads to $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$, imply that $\tau\,(C) > \tau(A|a) \simeq \tau(B|b)$. Indeed, for the strategy profile implying the largest value of $\tau(A|a)$ and $\tau(B|b)$ conditional on $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$, i.e. $\sigma(\varnothing|t_A) = 0 = \sigma(\varnothing|t_B)$, $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \frac{r(t_B|b)+r(t_A|b)}{r(t_A|a)+r(t_A|b)}$, and $\sigma(B|t_A) = 1 - \frac{r(t_B|b)+r(t_A|b)}{r(t_A|a)+r(t_A|b)}$, we have $\tau\,(C) > \tau(A|a) \simeq \tau(B|b)$. Then, the pivot probabilities between A and C in state a and between B and C in state b become infinitely larger than any other pivot probability, by Property 2. The payoffs $G\,(A|t)$ and $G\,(B|t)$ are both larger than zero. Abstention is therefore a dominated strategy.

Following Theorem 2 of Myerson 1998, note that if a type $t \in \{t_A, t_B\}$ adopts a strictly mixed strategy, then the other type $t' \neq t$, $t' \in \{t_A, t_B\}$ votes for "his" candidate with probability 1. The reason is that the priors $q\,(a|t)$ and $q\,(b|t)$ are different across types, which implies $G\,(A|t_A) - G\,(B|t_A) > G\,(A|t_B) - G\,(B|t_B)$ for any expected voting profile.

Having noted this, we know that the strategy profile leading to $\tau(A|a) \simeq \tau(B|b)$ is an equilibrium if, for that ranking of the vote shares, we have

$$\begin{aligned} &G\,(A|t_A) - G\,(B|t_A) \geq 0, \text{ and} \\ &G\,(A|t_B) - G\,(B|t_B) \leq 0. \end{aligned} \tag{12}$$

with at least one strict inequality. That is, types t_A must be willing to support A, and conversely for types t_B. Using (4), it is immediate to check that these inequalities hold iff $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$.

Next, remark that: *a)* pivot probabilities are continuous in $\sigma(A|t)$ and $\sigma(B|t)$, and *b)* payoffs are bounded, which allows us to apply Kakutani's fixed point theorem on $G(A|t) - G(B|t)$.

If voters marginally increase their propensity to vote A above the point in which $\tau(A|a) \simeq \tau(B|b)$, we have: $\tau(A|a) > \tau(B|b) > \tau(A|b) > \tau(B|a)$ and

$$\begin{aligned} G(A|t) - G(B|t) &> 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(A|a) \simeq \tau(B|b) > \tau(C) \\ G(A|t) - G(B|t) &< 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(A|a) \simeq \tau(B|b) < \tau(C), \end{aligned}$$

and the inequalities are reversed if the voters' propensity to vote for B increases. Two conclusions follow: *i) existence:* there must exist a strategy profile in the neighborhood of $\tau(A|a) = \tau(B|b)$ such that (12) holds. *ii) uniqueness:* $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$ is a necessary condition for the non-Duvergerian equilibrium.

<u>Step 2. If $\tau(C) > 1/[2 + r(t_A|b)/r(t_A|a)]$, the equilibrium is neither Condorcet-like nor stable</u>

As stated in step 1, if $\tau(C) > 1/[2 + r(t_A|b)/r(t_A|a)]$, all the strategy profiles that leads to $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$, imply that $\tau(C) > \tau(A|a) \simeq \tau(B|b)$. Hence, (5) is not satisfied and thus the equilibrium is not Condorcet-like. ■

$\tau(C) > \tau(A|a) \simeq \tau(B|b)$ also implies that the difference $G(A|t) - G(B|t)$ converges to:

$$\lim_{n\to\infty} G(A|t) - G(B|t) = q(a|t)\ \ 2\Pr(piv_{AC}|a) - q(b|t)\ \ 2\Pr(piv_{BC}|b).$$

Then, if the vote share of A increases (the argument is symmetric if it decreases), $\tau(C) - \tau(A|a)$ decreases, and $\tau(C) - \tau(B|b)$ increases. This increases the magnitude of $\Pr(piv_{AC}|a)$ and decreases that of $\Pr(piv_{BC}|b)$. Hence, the initial rise in $\tau(A|a)$ induces the value of a vote for A to increase, for both types t_A and t_B. The equilibrium is thus not "stable" in the sense that an exogenous increase in the vote share of A induces voters to increase their propensity to vote for A.

FIGURES

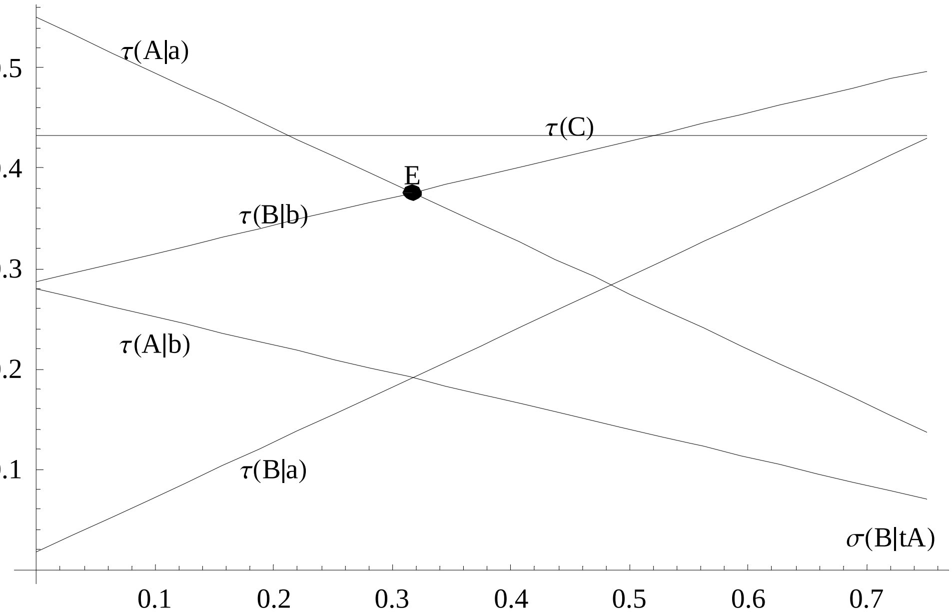

Figure 1. Knife edge equilibrium for $\tau\left(C\right)$ large

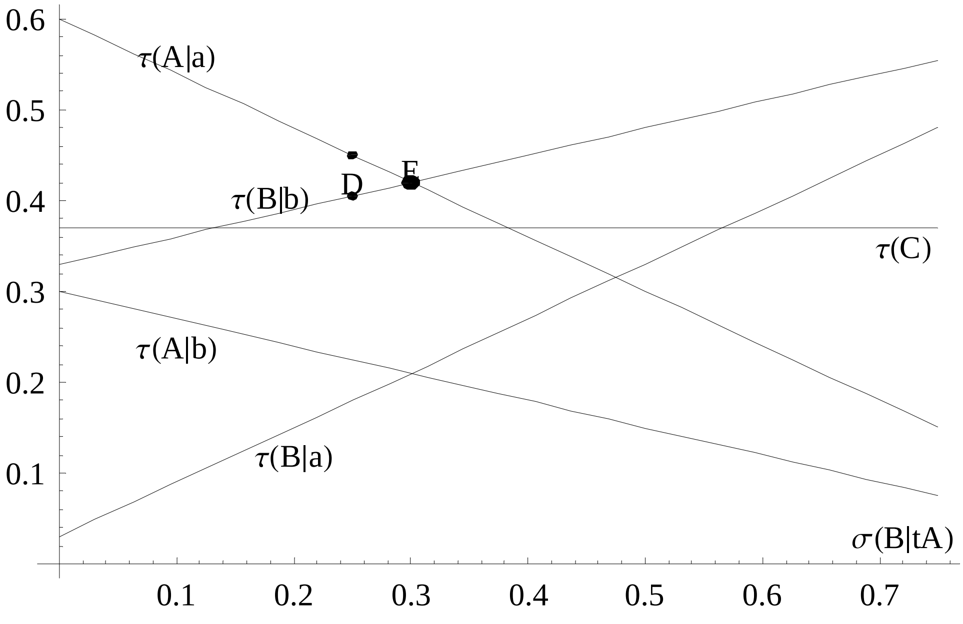

Figure 2. Stable equilibrium for $\tau\left(C\right)$ small

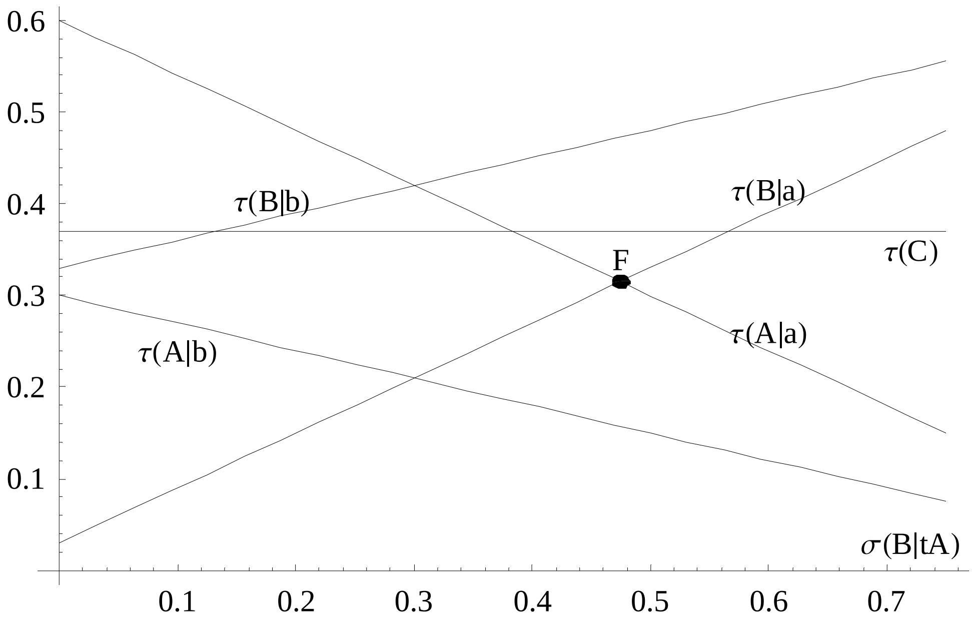

Figure 3. New type of equilibrium for $\tau\,(C)$ small.