One Person, Many Votes: Divided Majority and Information Aggregation[*]

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Abstract

This paper shows that information imperfections and common values can dramatically affect the properties of electoral systems. We analyze a three-alternative election in which the majority is divided between two alternatives. These alternatives compete with a third one that the majority considers as strictly inferior. Standard analyses assume that voters have fixed preference orderings over candidates. In that setup, no electoral system allows the divided majority to pick the best alternative for sure. In contrast, our analysis brings insights from the Condorcet Jury Theorem into the analysis of electoral systems. We show that when the common value component is relevant and the size of the electorate sufficiently large, Approval Voting produces a unique equilibrium in which the full information Condorcet winner is elected with a probability that approaches one.

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1 Introduction

Elections are fraught with uncertainty. Voters know little about the value of each candidate's platform or intrinsic merit. They are thus swift to react upon a breaking news: fresh information may create massive swings in electoral support. Think for instance of J.M. Aznar's lie when he blamed ETA for Al Qaeda's 2004 bombing in Madrid.[1] Or think about corruption charges: Ferraz and Finnan (2008) identify a causal link between public audits of local government, which reveals possible corruption misdemeanors, and a 20 percent drop in reelection probabilities in Brazil.[2] Swings in support also happens with common policy choices: think for instance of financial sector deregulation or the war on terror. Both policies had hidden costs that critically weakened the Republican presidential candidate John McCain in 2008. In all these cases, the voters' preference for one or another alternative is driven by their desire to select objectively better policies or candidates. That is, voters share common values along some policy dimensions.

In this paper, we show that the presence of information imperfections and common values can dramatically affect the properties of electoral systems because voters may wish to rely on each others' information to identify the best candidate. While this wish is overwhelmed by other strategic considerations in standard electoral systems such as Plurality,[3] we show that it leads to a unique and efficient equilibrium outcome under the electoral system known as *Approval Voting*, which relieves voters from the constraint of having to vote for only one candidate.[4] More precisely, we show that under Approval voting, the winner of the election is the same as if voters could access all the information available and coordinate their votes on the alternative that, in light of this information, is preferred by the majority. We call this property Full Information and Coordination Equivalence.

So far, the analyses of electoral systems have typically overlooked this type of information imperfections and the presence of common values. The classical approach is to focus on voters having a fixed preference ordering over candidates, which implies that no additional element of information may influence voter support (see e.g. Arrow 1951, Myerson and

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[1] J.M. Aznar (right-wing) was Spain's prime minister at the time and ran very high in opinion polls. In a bomb attack, Al Qaeda killed hundreds of persons in Madrid three days before the election. Despite the evidence, J.M. Aznar argued that the ETA was behind the attack. The revelation that he voluntarily tried to mislead the electorate led to a massive reversal in voters' support and led to the surprise victory of the socialist party.

[2] See also Peters and Welch (1980).

[3] See Bouton and Castanheira (2009).

[4] Under Approval Voting, voters can "approve of" as many candidates as they want, each approval counts as one vote and the candidate that obtains the largest number of votes wins (Weber 1977, 1995, Brams and Fishburn 1978, 1983).

Weber 1993, Fey 1997, Myerson 2002, Bag et al. 2008). Alternatively, the Condorcet Jury Theorem literature analyzes the effect of imperfect information but it focuses on two-candidate elections (see e.g. Austen-Smith and Banks 1996, Feddersen and Pesendorfer 1996, 1997, 1999, Myerson 1998, Kim and Fey 2007, Bhattacharya 2007), while one must consider at least three candidates to understand the strengths and weaknesses of most electoral systems.[5] The novelty of our analysis is to bring insights of the latter literature into the analysis of electoral systems and show that some perceived weaknesses of Approval Voting hinge upon the assumption of fixed preferences.

While an analysis of electoral systems for all possible situations is beyond the scope of this paper, these results do show that our knowledge of electoral systems is limited to a particular framework of analysis. Take the electoral system we study: what we know is that, with fully predetermined voter preferences, Approval Voting is (i) riddled with multiple equilibria, (ii) a Condorcet winner is not sure to win, and (iii) even a Condorcet loser may have significant chances of winning the election (see e.g. Myerson and Weber 1993, De Sinopoli *et al.* 2006 or Nuñez 2010). We show instead that when information is imperfect and the common value component is relevant, the equilibrium outcome is unique, and the full information Condorcet winner wins with a probability that approaches one as electorate size increases.

This resolves a central problem in the analysis of multicandidate elections: the *problem of the divided majority*, that dates back at least to Borda (1781). It arises when a majority of the electorate is divided between two alternatives, while a unified minority backs a third alternative. This division typically reinforces the minority candidate which is the reason for equilibrium multiplicity and the existence of inferior outcomes (see e.g. Myerson and Weber 1993, Piketty 2000, Myatt 2007, Dewan and Myatt 2007). Our setup focuses exactly on this situation: we analyze a three-alternative election in which the majority is divided as to which of two alternatives, A or B, is best. These two alternatives face competition from a third alternative, C, that the majority considers as strictly inferior.[6] In contrast to traditional formalization, we introduce information imperfections and common values: some majority voters *believe* that the best candidate is A; others *believe* it is B. Yet, all realize that they may be wrong. It is this doubt that gives majority voters an incentive to also rely on the information held by the rest of the electorate under Approval Voting. This incentive in turn produces full information and coordination equivalence.

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[5] A notable exception is Martinelli (2002), that studies two-round electoral systems and finds that information can be aggregated efficiently. But Bouton (2008) proves the existence of other equilibria with no information aggregation whenever there is uncertainty in the second round.

[6] Thus, our results do not hinge on dichotomous preferences, which would assume that each voter is indifferent between the "bottom two" alternatives.

Importantly, this equivalence does not hold in the standard first-past-the-post system, in which majority voters must beforehand select which of A or B to support: this is why information aggregation fails in that system. Under Approval Voting instead, majority voters can also multiple-vote. In particular, they can approve of both A and B, each approval counting as one vote. We show that, in equilibrium, some majority voters double vote to ensure that both A and B collect more votes than C, whereas other majority voters single-vote for their preferred alternative to produce information aggregation. For a sufficiently large electorate size, this equilibrium is unique and ensures that, depending on the true state of nature, either A or B wins the election. These results are proved in a setup with two states of nature and purely common values (Sections 2 to 4), as well as with infinitely many states of nature and a combination of private and common values (Section 5).

2 The Model

This section lays out our base model, with Poisson distributions, two states of nature, and voters in the majority having purely common value preferences.[7]

There are three alternatives, indexed by $P \in \{A, B, C\}$, two states of nature, $\omega \in \{a, b\}$, and three types of voters, $t \in T \equiv \{t_A, t_B, t_C\}$. Conditional on the state of nature, types t_A and t_B hold identical preferences: they always want to elect the best alternative, which is A in state a and B in state b:

$$\begin{aligned} U(P, t_A, \omega) = U(P, t_B, \omega) &= 1 \text{ if } (P, \omega) = (A, a) \text{ or } (B, b) \\ &= 0 \text{ if } (P, \omega) = (A, b) \text{ or } (B, a) \\ &= -1 \text{ if } P = C, \end{aligned} \tag{1}$$

where $U(P, t, \omega)$ denotes the utility of a voter with type t when alternative P is elected and the true state is ω. The values 1, 0 and -1 are only meant to simplify exposition.

Yet, from an *ex ante* vantage point, types t_A and t_B have opposite convictions regarding alternatives A and B: given their private signal (see below), they hold opposite beliefs as to which state is most likely. A voter with type t believes that the true state is ω with a probability $q(\omega|t)$. We impose that:

$$\infty > \frac{q(a|t_A)}{q(b|t_A)} > 1 > \frac{q(a|t_B)}{q(b|t_B)} > 0. \tag{2}$$

That is, *ex ante*, types t_A strictly prefer A to B and conversely for types t_B. Yet, these voters' information is imperfect: types t_A do not put a probability 1 on the true state being

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[7] Section 5 extends the model to the presence of private values and an infinite number of states of nature, and to multinomial distributions.

a (otherwise the probability ratio would be infinite), and types t_B do not put a probability 1 on the true state being b (otherwise the probability ratio would be zero). Yet, these priors may be arbitrarily close to 1. The relevant difference between priors being close or equal to 1 is that the voters' beliefs can change through Bayesian updating if other voters reveal some of their information.

For the sake of simplicity, we assume that types t_C are pure partisans: they always vote for alternative C.[8]

Timing. At the beginning of the game (**time 0**), nature chooses the state ω, which remains unobserved until after the election. The probabilities of states a and b are common knowledge, and denoted by $q(a)$ and $q(b)$, with $q(a) + q(b) = 1$. At **time 1**, nature selects a random number of voters from a Poisson distribution of mean n and, conditional on the state, assigns them a type t by iid draws.[9] The conditional probability of being assigned type t is $r(t|\omega)$, with $\sum_t r(t|\omega) = 1$, $\forall\omega$. These probabilities correlate with the true state of nature:

$$\begin{aligned} r(t_A|a) &> r(t_A|b), \\ r(t_B|a) &< r(t_B|b), \\ r(t_C|a) &= r(t_C|b), \end{aligned}$$

and are also common knowledge. To ensure that our results cannot hinge on any type of symmetry across types t_A and t_B, we allow types t_A to be potentially more "abundant" than t_B:

$$r(t_A|a) + r(t_A|b) \geq r(t_B|a) + r(t_B|b).$$

The distribution of voters determines which type is expected to be the majority. We focus on the case:

$$r(t_C|\omega) < 1/2, \tag{3}$$

which implies that, in expected terms, types t_C are a strict minority.[10] Hence, types t_A and t_B compose the *majority block*, whereas types t_C form the *minority block*.

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[8] Since our focus in on how elections may (or not) aggregate information despite the presence of a third party, this assumption actually maximizes the deleterious influence of alternative C: if types t_C were also voting for A and/or B, the hurdle faced by the other types would be lower.

[9] Section 5.3, shows that the Poisson assumption is not central to the results.

[10] For $r(t_C|\omega) > 1/2$, an expected majority of the electorate prefers that C wins, independently of ω. This case is trivial to investigate: by the law of large numbers, the realized fraction of types t_C will be larger than 1/2 with a probability that converges to 1 as $n \to \infty$ and C then necessarily wins.

The election is held at **time 2**. Neither the actual state of nature nor the actual number of voters of each type is observed: voters only know their own type, t.[11] Through Bayesian updating, a voter with type t infers that the probability of state ω is $q(\omega|t)$:

$$q(\omega|t) = \frac{q(\omega)\, r(t|\omega)}{q(a)\, r(t|a) + q(b)\, r(t|b)}, \tag{4}$$

and clearly, condition (2) imposes restrictions on $q(\omega)$ and $r(t|\omega)$.

Payoffs realize at **time 3**: the winning alternative $W \in \{A, B, C\}$ is selected and each voter receives utility $U(W, t, \omega)$.

Action set under Approval Voting. Under *Approval Voting*, each voter can cast a ballot on as many (or as few) alternatives as she wishes. Each approval counts as one vote: when a voter only approves of A, then only alternative A is credited with one vote. If the voter approves of both A and B, then both A and B are credited with one vote, and so on. Hence, the voters' action set is:

$$\Psi = \{A, B, C, AB, AC, BC, ABC, \varnothing\},$$

where, by an abuse of notation, action A denotes a ballot in favor of A only, action BC denotes a joint approval of B and C, etc., and $\varnothing$ denotes abstention. Thus, the difference between approval voting and other, more common, electoral rules is that a voter can cast a single, a double or a triple approval. Single approvals ($\psi = A$, B and C) act as positive votes: for instance, an A-vote can only be pivotal in favor of A, either against B or against C. In a three-alternative setup, double approvals ($\psi = AB$, BC and AC) act as negative votes. For instance, if the voter plays AC, she is acting against B: her ballot can only be pivotal against that alternative, either in favor of A or of C. Finally, a triple approval (ABC) can never be pivotal: it is strategically equivalent to abstention.

Let $x(\psi)$ denote the number of voters who played action $\psi \in \Psi$ at time 2. Then, the *total number of approvals* received by alternatives A, B, and C are respectively:

$$\begin{aligned} X(A) &= x(A) + x(AB) + x(AC) + x(ABC), \\ X(B) &= x(B) + x(AB) + x(BC) + x(ABC), \\ X(C) &= x(C) + x(AC) + x(BC) + x(ABC). \end{aligned} \tag{5}$$

The alternative with the largest total number of approvals wins the election. Ties are resolved by the toss of a fair coin.

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[11] When all majority voters have the same initial prior $q(\omega)$, a voter's type is uniquely defined by the signal she receives (t_A or t_B). Our results however directly extend to any other values of $q(\omega|t)$ that satisfies (2).

Strategy space and equilibrium concept. A type t's *strategy function* is a mapping $\sigma(t) : T \to [0,1]^8$ where $\sigma(\psi|t)$ denotes the probability that a randomly sampled voter of type t plays action ψ, and the usual constraint applies: $\sum_{\psi} \sigma(\psi|t) = 1$, $\forall t$. This strategy function $\sigma(t)$ reflects the fact that a voter can only condition her strategy on her type t. Given $\sigma(t)$, a fraction:

$$\tau(\psi|\omega,\sigma) = \sum_{t} r(t|\omega)\ \ \sigma(\psi|t) \tag{6}$$

of the electorate is expected to play action ψ in state ω. We call $\tau(\psi|\omega,\sigma)$ the *expected share of voters* who choose action ψ in state ω given the strategy σ. The expected number of ballots ψ is therefore:

$$\mathsf{E}[x(\psi)|\omega,\sigma] = \tau(\psi|\omega,\sigma) \cdot n.$$

Of course, the *realized* number of such ballots, $x(\psi)$, is random and its distribution depends on the strategy function, through $\tau(\psi|\omega,\sigma)$, and on the distribution of voters, which is Poisson in the core model, and multinomial in Section 5.3.

For this voting game, we analyze the limiting properties of symmetric Bayesian Nash equilibria when the expected population size n becomes infinitely large.[12] As shown by Myerson (2000, Theorem 0), at least one strategy function must be an equilibrium of this game. In such an equilibrium, each voter plays an undominated strategy given the expected vote shares of each alternative, and vote shares in turn result from this strategy.

3 Poisson games, Payoffs, Undominated Strategies, and Informational Traps

This section begins with a reminder of some elementary properties of Poisson games, which we extend to Approval Voting. Second, we identify the voters' expected payoffs and show that among the eight actions in Ψ, only three are undominated. Third, we show that the election necessarily reveals information about the true state of nature in equilibrium.

3.1 Large Poisson games in Approval Voting

In a Poisson game, population size follows a Poisson distribution of mean n. Since types are attributed by iid draws, the number of voters of each type also follows a Poisson distribution,

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[12] Note that the equilibrium mapping $\sigma(\psi|t)$ *must* be identical for all voters of a same type t, by the very nature of population uncertainty (see Myerson 1998b, p377, for more detail). Section 5.1 extends the model to a continuum of types, in which case the equilibrium is in cutoff strategies.

of mean $r(t|\omega)\ n$, and, as shown by Myerson (2000), the number of ψ-votes follows a Poisson distribution of mean $\tau(\psi|\omega,\sigma)\ n$:

$$\Pr(x(\psi)|\omega,\sigma) = \exp(-\tau(\psi|\omega,\sigma)\ n)\ \tfrac{(\tau(\psi|\omega,\sigma)\ n)^{x(\psi)}}{x(\psi)!}. \tag{7}$$

From (7), the probability that two actions, ψ_1 and ψ_2 are played the same number of times is:

$$\Pr(x(\psi_1) = x(\psi_2)|\omega) = \prod_{\psi\in\{\psi_1,\psi_2\}} \left(\exp(-\tau(\psi|\omega,\sigma)\ n)\ \sum_{k=0}^{\infty} \frac{(\tau(\psi|\omega,\sigma)\ n)^k}{k!}\right). \tag{8}$$

For the sake of readability, we henceforth omit σ from the notation and simply write $\tau(\psi|\omega)$. As shown by Myerson (2000, Theorems 1 and 2, and Corollary 1), it follows from (8) that:

Property 1 *For a large population of size n and for $c << n$, since $x(\psi|\omega)$ and $x(\psi'|\omega)$ are independent Poisson distributions:*

$$\mu(\psi,\psi') \equiv \lim_{n\to\infty} \frac{\log\left[\Pr\left(x(\psi) = x(\psi') \pm c|\omega\right)\right]}{n} = -\left(\sqrt{\tau(\psi|\omega)} - \sqrt{\tau(\psi'|\omega)}\right)^2.$$

That is, the probability that the number of ψ and ψ' votes do not differ by more than $c \in \mathbb{N}$ is exponentially decreasing in n. $\mu(\psi,\psi') \in [-1,0]$ is called the magnitude *of that event. Its absolute value represents the "speed" at which the probability decreases towards 0: the bigger the difference in the expected vote shares of ψ and ψ', the more negative is the magnitude (and the faster the probability goes to 0).*

Myerson (2000, p18) also shows that:

Property 2 *Compare two events with different magnitudes: $\mu(\psi_1,\psi_2) < \mu(\psi_3,\psi_4)$. Then, the probability ratio of the former over the latter event goes to zero as n increases:*

$$\mu(\psi_1,\psi_2) < \mu(\psi_3,\psi_4) \Longrightarrow \frac{\Pr(x(\psi_1) = x(\psi_2)|\omega)}{\Pr(x(\psi_3) = x(\psi_4)|\omega)} \underset{n\to\infty}{\longrightarrow} 0.$$

The issue is that these two properties are not sufficient to pinpoint pivot probabilities under approval voting. Indeed, voters can vote for more than one alternative, which introduces a correlation between vote results. By contrast, we need to rely on statistically independent Poisson variables to apply Property 1. Our strategy[13] is to keep track of each different action ψ: let us consider the event that alternatives A and B receive the same

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[13] In independently developed research, Nuñez (2009) follows a similar strategy to pinpoint pivot probabilities under Approval Voting.

number of approvals (the reasoning extends immediately to any other pair of alternatives). By (5):

$$X(A) = X(B) \text{ iff } X(A \backslash B) \equiv x(A) + x(AC) = x(B) + x(BC) \equiv X(B \backslash A).$$

Clearly, $X(A \backslash B)$ and $X(B \backslash A)$ are statistically independent. By the properties of Poisson distributions these are two Poisson distributions of mean $[\tau(A|\omega) + \tau(AC|\omega)]n$ and $[\tau(B|\omega) + \tau(BC|\omega)]n$ respectively. Thus:

Lemma 1 *The magnitude of the event that alternatives A and B have the same number of votes is:*

$$\mu(A,B) \equiv -\left(\sqrt{\tau(A|\omega) + \tau(AC|\omega)} - \sqrt{\tau(B|\omega) + \tau(BC|\omega)}\right)^2.$$

$\mu(A,C)$ and $\mu(B,C)$ are characterized equivalently.

Proof. See Appendix A1. ■

Now that we have characterized the magnitude of such events, we are in a position to derive the probability that a single vote is *pivotal* between two alternatives. Being *pivotal* means that a single ballot changes the identity of the winner, *i.e.* of the alternative with the largest number of votes. For instance, if $X(A) = X(B) > X(C)$, both A and B win the election with probability $\frac{1}{2}$. An additional ballot A (or AC) changes the outcome to $X(A) > \max[X(B), X(C)]$, in which A wins with probability 1. By contrast, if $X(C) - 1 > X(A) = X(B)$ an additional ballot A (or AC) does not affect the outcome of the election.

Let piv_{PQ} denote the event of being *pivotal between alternatives P and Q*. The probability of piv_{PQ} depends on the joint realization of two events: (i) either P and Q have the same number of votes or P trails behind Q by exactly one vote, and (ii) the third alternative R does not have more votes than Q. This pivot probability turns out to depend on the expected ranking of the candidates:

Lemma 2 *For any triplet of alternatives P, Q, $R \in \{A, B, C\}$, let:*

$$mag(piv_{PQ}|\omega) \equiv \lim_{n\to\infty} \frac{1}{n} \log\left[\Pr\left(X(Q) - X(P) \in [0,1] \ \& \ X(Q) \geq X(R)\,|\omega\right)\right].$$

Then:

$$\begin{aligned} mag(piv_{PQ}|\omega) &= \mu(P,Q) \text{ if } P \text{ and } Q \text{ have the largest two expected vote shares,} \\ &< \mu(P,Q) \text{ if } P \text{ and } Q \text{ have the lowest two expected vote shares.} \end{aligned}$$

Thus, if $\mathsf{E}[X(P)|\omega] > \mathsf{E}[X(Q)|\omega] > \mathsf{E}[X(R)|\omega]$:

$$\lim_{n\to\infty} \Pr(piv_{QR}|\omega) / \Pr(piv_{PQ}|\omega) = 0.$$

Proof. See Appendix A1. ■

As detailed in the proof of Lemma 2, the magnitude of the pivot probability between P and Q is necessarily *restricted*, and thus smaller, when alternative R has a higher expected vote share than Q. By Property 2, the pivot probability of a *restricted event* becomes infinitely smaller than the probability of any event with *unrestricted* magnitude as n increases.

Remark 1 *The correlation introduced by double voting implies that the largest magnitude need not be between the top two alternatives: for a given difference in expected vote shares, if two alternatives are more correlated through double voting, the probability of being pivotal between them is reduced. Thus, the largest magnitude can be the one between the alternatives that rank first and third in terms of expected vote shares.*

3.2 Payoffs and undominated strategies

Denoting by $\Pr(P \text{ wins}|\omega)$ the probability that alternative $P \in \{A, B, C\}$ wins the election in state ω, the expected utility of a majority-block voter $t \in \{t_A, t_B\}$ is:

$$\mathsf{E}U(t) = q(a|t)\ [\Pr(A \text{ wins}|a) - \Pr(C \text{ wins}|a)] + q(b|t)\ [\Pr(B \text{ wins}|b) - \Pr(C \text{ wins}|b)],$$

which reads as follows: having observed her type t, the voter anticipates that the true state of nature is a with probability $q(a|t)$. In that case, by (1), her utility is 1 if A wins, 0 if B wins, and -1 if C wins. With probability $q(b|t) \equiv [1 - q(a|t)]$ the true state is b. In that case, her payoff is 0 if A wins, 1 if B wins, and -1 if C wins.

A ballot affects the probability that each alternative wins in each state of nature. The value of a ballot value thus depends on its probability of being pivotal in each of these two states of nature. A ballot approving of two or more alternatives can never be pivotal between these. Otherwise, all pivot probabilities are strictly positive in a Poisson game. This directly implies that:

Lemma 3 *For a majority-block voter $t \in \{t_A, t_B\}$, in equilibrium:*

$$\sigma(A|t) + \sigma(B|t) + \sigma(AB|t) = 1, \tag{9}$$

since the other actions $\psi \in \{C, AC, BC, ABC, \varnothing\}$ are strictly dominated.

The proof is straightforward: consider a majority-block voter and compare actions AB and ABC. While the latter can never be pivotal, an AB-ballot can be pivotal against C, either in favor of A or in favor of B. In both cases, the majority-block voter utility increases.

Hence, AB strictly dominates ABC. All other strict dominance relationships are obtained by performing similar two-by-two comparisons: AB strictly dominates ABC, $\varnothing$ and C; A strictly dominates AC; and B strictly dominates BC. A corollary is that:

Corollary 1 *Abstention is a strictly dominated strategy: with three candidates, there is no swing voter's curse under approval voting.*

We can thus focus on these undominated actions. Let $G(\psi|t)$ denote the *expected gain* of action $\psi \in \{A, B, AB\}$ over abstention, $\varnothing$:

$$\begin{aligned} G(A|t) &= q(a|t)\ \left[\Pr(piv_{AB}|a) + 2\Pr(piv_{AC}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{AC}|b) - \Pr(piv_{AB}|b)\right], \end{aligned} \tag{10}$$

$$\begin{aligned} G(B|t) &= q(a|t)\ \left[\Pr(piv_{BC}|a) - \Pr(piv_{BA}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{BA}|b) + 2\Pr(piv_{BC}|b)\right], \end{aligned} \tag{11}$$

$$\begin{aligned} \text{and } G(AB|t) &= q(a|t)\ \left[\Pr(piv_{BC}|a) + 2\Pr(piv_{AC}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{AC}|b) + 2\Pr(piv_{BC}|b)\right]. \end{aligned} \tag{12}$$

These gains depend on the voter's type, via the perceived probability of each state $q(\omega|t)$, and on the pivot probabilities in each state, $\Pr(piv_{PQ}|\omega)$. As seen in the previous subsection, these pivot probabilities depend on the strategy function $\sigma(t)$, but we omit it from the notation for the sake of readability.

3.3 No informational trap under approval voting

If vote shares (and thus pivot probabilities) are the same in both states of nature, then the election result cannot generate any information about the true state of nature. This would necessarily prevent information aggregation. We say that:

Definition 1 *A strategy function σ^{IT} produces an **informational trap** if the expected result of the election is independent of the state of nature:*

$$\mathsf{E}\left(X(P)\,|a,\sigma^{IT}\right) = \mathsf{E}\left(X(P)\,|b,\sigma^{IT}\right),\ \forall P \in \{A, B, C\}.$$

By contrast, if vote shares (and thus pivot probabilities) differ across states of nature, the expected result of the election must reveal information about the true state of nature. Our first proposition shows that an informational trap can only occur for one particular strategy function:

Proposition 1 *If the strategy function* σ^{IT} *produces an informational trap, then:*
(i) All majority voters must adopt the same strategy: $\sigma^{IT}(t_A) = \sigma^{IT}(t_B)$*;*
(ii) $G\left(AB|t_A, \sigma^{IT}\right) > G\left(B|t_A, \sigma^{IT}\right)$ *and* $G\left(AB|t_B, \sigma^{IT}\right) > G\left(A|t_B, \sigma^{IT}\right)$ *: that is, it is never a best response for a type* t_A *(resp.* t_B*) to play* B *(resp.* A*);*
(iii) The only candidate informational trap in undominated strategies is thus with: $\sigma(AB|t_A) = 1 = \sigma(AB|t_B)$.

Proof. See Appendix A1. ■

The intuition for the proof is that, since the distribution of voters changes across states of nature, informational traps can only be supported by a strategy that is independent of the voter's type. Yet, when voters do not expect the election to elicit additional information, they want to play a "sincere strategy": they always approve of their *a priori* preferred alternative. The third point in the proposition follows from the first two: since voters always approve of their preferred alternative, the only strategy they could share is to approve of both A and B with probability 1. In that case, A and B necessarily tie.

This strategy function has been termed the *Burr dilemma* by Nagel (2007), who argues that approval voting is inherently biased towards such ties. He documents this with the "[approval] *experiment* [that] *ended disastrously in 1800 with the infamous Electoral College tie between Jefferson and Burr*". Our second proposition however shows that such a strategy cannot be an equilibrium when the electorate size is sufficiently large:

Proposition 2 *There exists a threshold* $\bar{n}$ *such that there is* no *equilibrium with an informational trap for* $n \geq \bar{n}$.

Proof. See Appendix A1. ■

The reason for this "no informational trap" result is that, as population size increases and if all majority voters double vote, the probability of being pivotal between A and B increases to 1/2 (remember that a fair coin is tossed to determine the winner in case of a tie), whereas the probability of being pivotal against C progressively decreases to 0. Indeed, $X(A)$ is necessarily equal to $X(B)$: they correlate perfectly. By contrast, since she is trailing behind, C's chance of winning goes to zero.

We term this the *information motive*: majority voters prefer to single vote for their preferred alternative if they are (almost) sure that C loses and the election outcome does not reveal additional information. That is, types t_A only approve of A and types t_B only of B. This information motive will be a recurrent force in the analysis; it prevents majority voters from double voting "excessively".

4 Approval Voting: Equilibrium Properties

This section identifies the properties of the equilibrium strategy in approval voting. In a first-best world, majority voters would like to aggregate all the elements of information in their possession, use this information to update their beliefs about the actual state of nature, and coordinate their votes on the best alternative (i.e. the full information Condorcet winner). In reality, which is the best alternative is unclear at the time of election, and coordination problems may arise. Our benchmark is that the outcome of the election should reproduce the first best with a probability that approaches 1 as population size increases:

Definition 2 *A strategy function σ satisfies* ***full information and coordination equivalence*** *if its associated vote shares are such that:*

$$\begin{aligned} &\tau(A|a) + \tau(AB|a) > \max\left\{\tau(B|a) + \tau(AB|a), \tau\left(C\right)\right\} \text{ in state } a, \text{ and} \\ &\tau(B|b) + \tau(AB|b) > \max\left\{\tau(A|b) + \tau(AB|b), \tau\left(C\right)\right\} \text{ in state } b. \end{aligned} \tag{13}$$

That is, alternative A's expected vote share is the largest one in state a and conversely for alternative B in state b. Asymptotically, the winning alternative is then the full information Condorcet winner.[14]

Satisfying full information and coordination equivalence (FICE) is not a trivial matter in a three-alternative election: first, C may win the election if the majority split their votes. Second, there cannot be FICE if all majority-block voters approve of either A or B. Third, as explained in the introduction, coordination issues arise when there are multiple equilibria: for instance, all majority block voters may want to approve of a same alternative but cannot agree on which of A or B to coordinate. Theorem 1 shows that these issues do not arise under Approval Voting when population size is large enough:

Theorem 1 (Full information and coordination equivalence) *Under Approval Voting, there exists an expected population size $\bar{n}$, such that for any $n \geq \bar{n}$ the equilibrium is unique and satisfies full information and coordination equivalence.*

When the majority is divided, all electoral systems are typically riddled with multiple equilibria. Theorem 1 thus contrasts with such a typical property. As we briefly explain here and demonstrate below, it is the presence of common value that produces uniqueness in Approval Voting (AV). Otherwise, multiple equilibria exist also under AV.

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[14] This concept of *full information and coordination equivalence* is a natural extension to multicandidate elections of Feddersen and Pesendorfer's (1997) concept of *full information equivalence.*

The characteristic of AV is that voters have the option to double vote. There are two reasons why voters may wish to exert that option. The "standard" reason is what we call the *coordination motive*: out of equilibrium, whenever majority voters expect C to be the major threat against either A or B, they all develop an incentive to (also) approve of the threatened alternative. As we show below, this implies that C necessarily ranks third in equilibrium.

The novel rationale for double voting is what we call the *common-value motive*: neither types t_A or t_B want A to win in state b nor B to win in state a. Thus, if A is deemed "too strong" in state b, types t_A want to double vote AB; and similarly for types t_B if B is "too strong" in state a.

This common-value motive materializes because voters understand that their private signal is imperfect, that is: $q(a|t_A) < 1$ and $q(b|t_B) < 1$. In that case, voters compare their probability of being pivotal between A and B across the two states of nature in $(10)-(12)$. As we demonstrate below, one of these two pivot probabilities becomes infinitely larger than the other when FICE does not hold. All majority voters would then develop an incentive to (also) approve of the weaker alternative. This actually implies that FICE must hold in equilibrium. Importantly, this result does not only hold in a pure common value setup. In Sections 5.1 and 5.2, we show how and when it extends to a setup that combines private and common values.

By contrast, the standard "pure private value setup" amounts to setting $q(a|t_A) = 1$ and $q(b|t_B) = 1$, which means that voters only consider one state in $(10)-(12)$. Like in Myerson and Weber (1993), majority voters would then maintain an incentive to keep their *a priori* preferred alternative ahead of the other. Two additional equilibria without FICE would then exist: one in which types t_A only approve of A (and types t_B double vote) and one in which types t_B only approve of B (and types t_A double vote). The equilibrium properties of approval voting thus feature a discontinuity when prior beliefs are close to or equal to 1 (we return to this in Section 5.2).

It is also important to note that the common value motive is not sufficient to produce uniqueness in the standard plurality electoral system. In that system if one alternative, say A, dominates in both states of nature, all majority voters realize that their only serious chance of being pivotal against C is to vote for A. There are thus two so-called "Duvergerian" equilibria in which only A or only B receive votes from majority voters, as in, e.g. Cox (1997), Piketty (2000) or Castanheira (2003).

The rest of this section proves Theorem 1. Each of the next three subsections focuses on one aspect of the proof: first, we prove in Proposition 3 that t_A-voters adopt a strictly mixed strategy between A and AB, whereas types t_B mix between B and AB. Second, we show that A must have the largest vote share in state a, and B must dominate in state b. Third, we show that, for n large enough, there is a unique equilibrium strategy, which balances the information, coordination and common-value motives.

4.1 Voters Specialize

The first step to prove Theorem 1 is to show that no majority voter would ever mix between actions A and B. We then show that types t_A and t_B respectively mix between actions A and AB, and B and AB:

Lemma 4 *In equilibrium, a given majority type, t_A or t_B, never mixes between actions A and B.*

Proof. See Appendix A2. ■

This result closely relates to the swing voter's curse (Feddersen and Pesendorfer 1996; Myerson 1998). The intuition is as follows: in a setup with two candidates and common valued voters (*i.e.* our setup without candidate C), voters want to avoid mixing between actions A and B because they fear being "mistakenly pivotal," for instance in favor of B against A when the actual state is a. In our three-candidate setup, voters avoid mixing between actions A and B for the same reason. Yet, because of the threat posed by C, voters do not either want to abstain. Approval voting allows them to hit two birds with one ballot: they can double vote, and play action AB. This ensures that they only abstain between A and B, while it maximizes their probability of being pivotal against C.

Proposition 3 *There exists an expected population size $\bar{n}$, such that for any $n \geq \bar{n}$, in equilibrium, $\sigma(A|t_A) + \sigma(AB|t_A) = 1$ and $\sigma(B|t_B) + \sigma(AB|t_B) = 1$ with $\sigma(A|t_A) > 0$ and $\sigma(B|t_B) > 0$. That is, majority-block voters strictly mix between their 'preferred alternative' and the joint AB approval.*

Proof. See Appendix A2. ■

Proposition 3 implies that majority-type voters always vote "sincerely". That is, they necessarily approve of their *a priori* preferred alternative: since she mixes between actions A and AB, a type t_A always approves of A. Hence, and perhaps surprisingly, the strategy of types t_A only influences the vote share of B: the more types t_A double-vote, the higher the expected vote share of B, but the expected vote share of A is left unchanged. Similarly, types t_B mix between B and AB, which always includes B. Hence, it is their strategy that determines the expected vote share of A.

The intuition of the proof of Proposition 3 is as follows: we first show that actions A and B are played with strictly positive probability in equilibrium. Since by Lemma 4 a given voter type cannot mix between actions A and B it follows that some type must specialize in playing A and another type in playing B. Types t_A are then identified as the ones playing A with strictly positive probability (they never play B), and conversely for types t_B.[15]

To illustrate this result and show why actions A and B must be played with strictly positive probability, consider the following numerical example: $r(t_A|a) = 0.55$ and $r(t_A|b) = 0.51$, $r(t_B|a) = 0.05$ and $r(t_B|b) = 0.09$, and $r(t_C) = 0.4$. Now imagine that no voter plays $\psi = A$. This is for instance the case when $\sigma(AB|t_A) = 1$, $\sigma(B|t_B) = 1$ and $\sigma(C|t_C) = 1$, *i.e.* types-t_A double vote AB, types-t_B single vote B and types-t_C single vote C. For this strategy function, 60% of the electorate approves of B and 40% approve of C, whereas alternative A lies in between, with 55% in state a and 51% in state b. To show that this strategy profile is *not* an equilibrium, we apply Lemmas 1 and 2 which reveal that the magnitudes are:

Magnitudes	state a	state b
$mag(piv_{AC})$	-0.062	-0.097
$mag(piv_{BC})$	-0.02	
$mag(piv_{AB})$	-0.05	-0.09

The largest magnitude is thus the one between B and C, which confirms that types-t_A strictly prefer to play $\psi = AB$ $(G(A|t_A) < G(AB|t_A)$, by (14) below). However, types-t_B also strictly prefer to play $\psi = AB$: they are much more likely to be pivotal against A in state a than in state b ($G(B|t_B) < G(AB|t_B)$ by (15) below). This leads to a contradiction: all types B want to deviate from the strategy function that we initially conjectured. This argument extends to any strategy function in which $\tau(A|\omega) = 0$, and a symmetric reasoning applies for $\tau(B|\omega) = 0$.

Formally, using the expected gain functions $(10) - (12)$, the voters' relevant comparison of payoffs is the following:

$$\begin{aligned} G(A|t_A) - G(AB|t_A) &= q(a|t_A)\ \left[\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)\right] \\ &\quad - q(b|t_A)\ \left[2\Pr(piv_{BC}|b) + \Pr(piv_{AB}|b)\right] \gtrless 0, \end{aligned} \tag{14}$$

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[15] In a private value setup, Brams and Fishburn (2007, Theorem 2.1) show that a voter always includes her most preferred alternative in her ballot. One aspect of Proposition 3 is to show how their Theorem extends to voters whose preference ordering is state-dependent.

$$\begin{aligned} G\left(B|t_B\right) - G\left(AB|t_B\right) \quad &= q\left(b|t_B\right) \ \left[\Pr\left(piv_{BA}|b\right) - \Pr\left(piv_{AC}|b\right)\right] \\ &\quad - q\left(a|t_B\right) \ \left[2\Pr\left(piv_{AC}|a\right) + \Pr\left(piv_{BA}|a\right)\right] \gtrless 0. \end{aligned} \tag{15}$$

From Proposition 3, types t_A and t_B must single-vote with positive probability in equilibrium, which implies that the differences $G\left(A|t_A\right) - G\left(AB|t_A\right)$ and $G\left(B|t_B\right) - G\left(AB|t_B\right)$ must be non-negative. Non-negativity is central to explain why the equilibrium is unique. In the next two subsections, we isolate each one of two components in these non-negativity conditions.

4.2 Full Information and Coordination Equivalence

Here, we focus on the common value motive for double voting, which is shown to ensure that A has a larger vote share than B in state a and conversely in state b.

By (14), a necessary condition to have $G\left(A|t_A\right) - G\left(AB|t_A\right) \geq 0$ is that $\Pr\left(piv_{AB}|a\right)$ be sufficiently large compared to the other pivot probabilities in (14). Similarly, a necessary condition to have $G\left(B|t_B\right) - G\left(AB|t_B\right) \geq 0$ is that $\Pr\left(piv_{BA}|b\right)$ be sufficiently large compared to the other pivot probabilities in (15). That is, from Property 2:

$$\begin{aligned} mag\left(piv_{AB}|a\right) &\geq \max\left\{mag\left(piv_{AB}|b\right), mag\left(piv_{BC}|a\right), mag\left(piv_{BC}|b\right)\right\}, \\ mag\left(piv_{BA}|b\right) &\geq \max\left\{mag\left(piv_{BA}|a\right), mag\left(piv_{AC}|a\right), mag\left(piv_{AC}|b\right)\right\}. \end{aligned} \tag{16}$$

The common value motive materializes in the fact that $mag\left(piv_{AB}|a\right)$ and $mag\left(piv_{BA}|b\right)$ must be equal for (16) to be satisfied. By Lemma 1, a necessary condition is that:

$$\begin{aligned} &\left(\sqrt{r\left(t_A|a\right) \cdot \sigma\left(A|t_A\right)} - \sqrt{r\left(t_B|a\right) \cdot \sigma\left(B|t_B\right)}\right)^2 = \\ &\qquad \left(\sqrt{r\left(t_A|b\right) \cdot \sigma\left(A|t_A\right)} - \sqrt{r\left(t_B|b\right) \cdot \sigma\left(B|t_B\right)}\right)^2. \end{aligned} \tag{17}$$

This condition depends on the two strategy profiles, $\sigma\left(A|t_A\right)$ and $\sigma\left(B|t_B\right)$. Yet, defining:

$$\rho \equiv \sigma\left(A|t_A\right) / \sigma\left(B|t_B\right),$$

one readily sees that condition (17) is satisfied iff:

$$\left|\sqrt{r\left(t_A|a\right) \cdot \rho} - \sqrt{r\left(t_B|a\right)}\right| = \left|\sqrt{r\left(t_B|b\right)} - \sqrt{r\left(t_A|b\right) \cdot \rho}\right|,$$

which has a unique solution in $\mathbb{R}^+$:

$$\rho^* = \left(\frac{\sqrt{r\left(t_B|a\right)} + \sqrt{r\left(t_B|b\right)}}{\sqrt{r\left(t_A|a\right)} + \sqrt{r\left(t_A|b\right)}}\right)^2. \tag{18}$$

This solution in turn implies: $\tau(A|a) > \tau(B|a)$ and $\tau(A|b) < \tau(B|b)$. By Lemma 2 and since there is no double voting involving alternative C, condition (16) cannot be satisfied if $\tau(C) > \tau(B|a)$ and/or $\tau(C) > \tau(A|b)$. Therefore, in equilibrium:

$$\begin{aligned} \tau(A|a) &> \tau(B|a) \geq \tau(C), \text{ and} \\ \tau(B|b) &> \tau(A|b) \geq \tau(C). \end{aligned}$$

We are now left with only one unknown variable: if we find the equilibrium probability $\sigma(B|t_B)$ with which types t_B single-vote in equilibrium, the value of $\sigma(A|t_A)$ follows immediately.

4.3 Equilibrium Uniqueness

In this section, we show that, because of the interaction between the coordination and the information motives, there is a unique value of $\sigma(B|t_B)$ (and hence of $\sigma(A|t_A)$, by (18)) that can be an equilibrium:

Proposition 4 *There exists an expected population size $\bar{n}$, such that for any $n \geq \bar{n}$, the equilibrium strategy is unique and such that, in the limit $n \to \infty$:*
i) $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \rho^*$ *iff, for this strategy profile,*

$$mag(piv_{AB}|a) = mag(piv_{AB}|b) \geq \max_{\omega} \{mag(piv_{AC}|\omega), mag(piv_{BC}|\omega)\}.$$

ii) *Otherwise,* $\sigma(B|t_B) = \bar{\sigma}$, $\sigma(A|t_A) = \rho^*\bar{\sigma}$ *with* $\bar{\sigma} \in (0,1)$ *such that:*

$$mag(piv_{AB}|a) = mag(piv_{AB}|b) = \max_{\omega} \{mag(piv_{AC}|\omega), mag(piv_{BC}|\omega)\}. \tag{19}$$

Proof. See Appendix A2. ■

The proof is rather straightforward: if there is "too much" double voting, both A's and B's vote shares become large as compared to that of C. The information motive dominates in that case: both types t_A and t_B strictly prefer to single-vote for their initially preferred alternative. This increases the vote gap between A and B in both states of nature and hence the precision of the voting signal. The only obstacle to furthering this gap is the threat posed by C: if (19) binds, then the coordination motive dominates. In other words, both types t_A and t_B prefer to double-vote with a sufficiently high probability to ensure that A and B remain sufficiently ahead of C. The equilibrium is reached at the unique value of $\sigma(B|t_B)$ for which the coordination motive balances the information motive – unless a corner solution is reached. The solution is unique because the perceived threat posed by C decreases monotonically with the fraction of voters who double-vote.

4.4 Numerical Examples

To provide a more concrete interpretation of the above results, this subsection proposes a set of numerical examples that focus on symmetric priors: $q\left(a\right)=\frac{1}{2}=q\left(b\right)$ and a symmetric distribution of types: $r\left(t_A|a\right)=r\left(t_B|b\right)$. Symmetry is only meant to simplify exposition: from (18) and Proposition 4, it imposes that $\sigma^*(A|t_A)=\sigma^*(B|t_B)$. We first illustrate the effect of the common value motive and then of the coordination motive. We also use these examples to illustrate that population sizes of 10 or 100 thousand voters are largely sufficient for our limit results to have bite.

Let $r(t_C)=0.4$, $r(t_A|a)=0.36$ and $r(t_A|b)=0.24$. With these parameter values the Condorcet loser, C, would asymptotically win if the majority single voted for their a priori preferred candidate. Vote shares would indeed be: $\tau\left(C\right)=0.4>\tau\left(A|a\right)=\tau\left(B|b\right)=0.36>\tau\left(A|b\right)=\tau\left(B|a\right)=0.24$. This implies that we are in case (*ii*) of Proposition 4, and that there must be some double-voting in equilibrium. The equilibrium strategy profile is $\sigma(AB|t_A)=0.57=\sigma(AB|t_B)$, which leads to the expected vote shares and magnitudes illustrated in Table 1.

Table 1: equilibrium vote shares (left) and magnitudes (right).[16]

Vote shares	state a	state b
A	0.497 (first)	0.445 (second)
B	0.445 (second)	0.497 (first)
C	0.4 (third)	0.4 (third)
Total	1.342	1.342

and

Magnitudes	state a	state b
$mag(piv_{AC}|\omega)$	-0.0052	(restricted)
$mag(piv_{BC}|\omega)$	(restricted)	-0.0052
$mag(piv_{AB}|\omega)$	-0.0052	-0.0052

As this example illustrates, double voting allows the majority to "inflate" the expected vote shares of both A and B above the share of C. This is why the sum of the three vote shares exceeds 100% of the population. It also illustrates that, with an internal solution, the magnitudes of the pivot probabilities between A and B are equal to the largest magnitudes against C.

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[16] The pivot probability between the second and third candidates is infinitely lower than the pivot probability between the first and second candidate. In the absence of a closed-form solution for these magnitudes, we cannot compute their exact value.

4.4.1 Deviations

To illustrate the effect of the common value motive, consider for a moment a deviation by types t_A who increase their probability of playing AB by 3 percentage points, such that $\sigma(AB|t_A) = 0.6$. The expected vote shares and magnitudes are now:

Table 2: common value motive - equilibrium vote shares (left) and magnitudes (right).

Vote shares	state a	state b
A	0.497 (first)	0.4452 (second)
B	0.456 (second)	0.504 (first)
C	0.4 (third)	0.4 (third)
Total	1.353	1.349

and

Magnitudes	state a	state b
$mag(piv_{AC}\|\omega)$	-0.0052	(restricted)
$mag(piv_{BC}\|\omega)$	(restricted)	-0.0060
$mag(piv_{AB}\|\omega)$	$\mathbf{-0.0034}$	-0.0070

The major magnitude is thus the one between A and B in state a. This implies that, for a population size of n =10'000, the probability ratios are $\frac{\Pr(piv_{AB}|a)}{\Pr(piv_{AB}|b)} \simeq 4 \times 10^{15}$ and $\frac{\Pr(piv_{AB}|a)}{\Pr(piv_{AC}|a)} \simeq 1 \times 10^{8}$. For a population size of 100'000, they are $\frac{\Pr(piv_{AB}|a)}{\Pr(piv_{AB}|b)} \simeq 2 \times 10^{156}$, $\frac{\Pr(piv_{AB}|a)}{\Pr(piv_{AC}|a)} \simeq 2 \times 10^{80}$. This means that, even if t_B voters initially believed that the true state is a with a probability as low as 10^{-8}, they prefer to double vote if population size is at least as large as 10'000 voters. Likewise, if their prior is as low as $q\,(a|t_B) = 10^{-80}$, a population of 100'000 voters is sufficient for the common value motive to dominate.

Our third example illustrates how the aggregate level of double-voting affects the relative importance of the information and coordination motives: consider a strategy function for which majority voters do not double-vote enough: $\sigma(AB|t_A) = 0.54 = \sigma(AB|t_B)$ (recall that $\sigma(AB|t_A) = 0.57 = \sigma(AB|t_B)$ in equilibrium). Table 3 shows the expected vote shares and magnitudes for this strategy function:

Table 3: coordination motive - equilibrium vote shares (left) and magnitudes (right).

Vote shares	state a	state b
A	0.4896 (first)	0.4344 (second)
B	0.4344 (second)	0.4896 (first)
C	0.4 (third)	0.4 (third)
Total	1.324	1.324

and

Magnitudes	state a	state b
$mag(piv_{AC}\|\omega)$	$\mathbf{-0.0045}$	(restricted)
$mag(piv_{BC}\|\omega)$	(restricted)	$\mathbf{-0.0045}$
$mag(piv_{AB}\|\omega)$	-0.0056	-0.0056

Here, all majority voters realize that C is the main threat against either majority alternative. To reduce C's probability of winning, both majority types prefer to jointly approve of A and B. In particular, when n =10'000, the probability ratios are $\frac{\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|\omega)} = \frac{\Pr(piv_{AC}|a)}{\Pr(piv_{AB}|\omega)} = 4 \times 10^4$. For n =100'000, these ratios increase to 5×10^{45}. The relative weight put by majority voters on coordinating their ballots against C thus rapidly swamps their information motive of making a difference between alternatives A and B. Clearly, the effect goes in the other direction if there is more double voting than in equilibrium: their information motive would then dominate, and they would prefer to double vote less.

4.4.2 Comparative statics

The above parameter values ensure the existence of an internal equilibrium. A corner solution would instead be reached if the fraction of types t_C were sufficiently low: C is then not a serious threat. Actually, in a symmetric setup, majority-block voters double-vote in equilibrium if and only if $r(t_C) > r(t_A|b) = r(t_B|a)$. For instance, with $r(t_C) = 0.25$, $r(t_A|a) = 0.45$ and $r(t_A|b) = 0.30$, the equilibrium is that all majority types single-vote for their preferred alternative.

This observation directly links to Brams and Fishburn's (2005) case study of the Institute of Electrical and Electronics Engineers (IEEE). In 1986, because of a split among the majority, the minority-backed candidate almost won the election for the presidency. This triggered the adoption of Approval Voting by the Institute. Subsequently, both majority divisions and minority size decreased, which induced the IEEE to revert to Plurality Voting. Arguably, the latter decision overlooks the option value of a double-vote:

> *According to the IEEE executive director* [...] '*few of our members were using* [multiple voting...]'. *Brams responded in an e-mail exchange (June 2, 2002) that since "candidates now can get on the ballot with 'relative ease'* [...] *the problem of multiple candidates* [...] *might actually be exacerbated ... and come back to haunt you [IEEE] some day"* (Brams and Fishburn 2005, p16).

It is also interesting to look at the effect of the quality of information on the equilibrium: surprisingly, better information induces *more* double-voting. The rationale is as follows: increasing $r(t_A|a)$ and decreasing $r(t_A|b)$ while holding $r(t_C)$ constant implies that the gap between the first and the second alternative's vote shares increases for a given strategy profile. The probability of being pivotal between A and B thus decreases, which heightens the relative importance of the coordination motive. To illustrate this, set $r(t_A|a) = 0.48$

and $r(t_A|b) = 0.12$ and keep $r(t_C) = 0.4$ as in the first example. We find that $\sigma(AB|t_A) = 0.8580 = \sigma(AB|t_B)$ in equilibrium, and hence:

Table 4: equilibrium vote shares (left) and magnitudes (right).

Vote shares	state a	state b
A	0.583 (first)	0.532 (second)
B	0.532 (second)	0.583 (first)
C	0.4 (third)	0.4 (third)
Total	1.5144	1.5144

and

Magnitudes	state a	state b
$mag(piv_{AC})$	-0.0172	(restricted)
$mag(piv_{BC})$	(restricted)	-0.0172
$mag(piv_{AB})$	-0.0172	-0.0172

Compared to the first example, the equilibrium ranking remains the same but there is more double-voting and pivot magnitudes are lower, which means that, for any n, the probability of a mistake, *i.e.* that A wins in state b or B wins in state a, decreases substantially.

5 Robustness

5.1 Private and common values: a doubt is enough

Throughout, we worked under the assumption of majority voters who have identical preferences but differing information. What would happen if they also had heterogeneous preferences? In line with Feddersen and Pesendorfer (1997), let us extend our basic setup in the following way: each majority voter i has now a specific utility function $U_i(P|\omega)$, with a voter-specific valuation for each alternative P. There is a continuum of states of nature, $\omega \in [0,1]$, the distribution of which is denoted by the CDF $F(\omega)$ with $F(\omega) = 0$, $\forall \omega < 0$, $F(\omega) = 1$, $\forall \omega > 1$, and $f(\omega) \equiv F'(\omega) > 0$, $\forall \omega \in [0,1]$. The private value component is such that, for any *interior* state $\omega \in (0,1)$, we have:

$$\begin{aligned} \exists i \text{ such that } U_i(A|\omega) - U_i(B|\omega) &> 0 \\ \exists j \text{ such that } U_j(A|\omega) - U_j(B|\omega) &< 0. \end{aligned}$$

We also impose that, for any $\delta > 0$ and $\omega \in (0,1)$, the fraction of voters with $|U_i(A|\omega) - U_i(B|\omega)| < \delta$ is strictly positive. Like in our basic setup, we maintain the assumption that $U_i(A,\omega) - U_i(C,\omega) > 0$ and $U_i(B|\omega) - U_i(C|\omega) > 0$, for any i, ω.

The common value component is represented by the fact that, for any $\omega > \omega'$, we have:

$$U_i(A|\omega) - U_i(B|\omega) > U_i(A|\omega') - U_i(B|\omega'),$$

and that all voters have the same preference ordering in the two *extreme* states:

$$\begin{aligned} U_i(A|1) - U_i(B|1) &> 0,\ \forall i, \\ U_i(A|0) - U_i(B|0) &< 0,\ \forall i. \end{aligned}$$

The *doubt* is introduced by setting $F(0^+) \equiv \lim_{\omega \searrow 0} F(\omega) > 0$ and $F(1^-) \equiv \lim_{\omega \nearrow 1} F(\omega) < 1$: both extreme states have strictly positive (possibly arbitrarily small) probability. Thus, for any voter i, there is a strictly positive (but possibly arbitrarily small) probability that she actually prefers A to B and conversely.

Prior to the election, each voter receives an independent and identically drawn signal $s \in \{0,1\}$. Each signal is received with probability $r(s|\omega) \in (0,1)$, with $r(1|\omega) > r(1|\omega')$ for any $\omega > \omega'$. That is, signal 1 is more likely when A is better valued. In line with the simple setup, we assume that the private signal is sufficiently informative, such that a strictly positive fraction of voters are "sensitive" to the signal:

$$\begin{aligned} \exists i \text{ s.t. } \int q(\omega|1)\left[U_i(A|\omega) - U_i(B|\omega)\right] dF(\omega) &> 0, \text{ and} \\ \int q(\omega|0)\left[U_i(A|\omega) - U_i(B|\omega)\right] dF(\omega) &< 0, \end{aligned}$$

where $q(\omega|s) = r(s|\omega)\ f(\omega)/r(s)$ is the belief about the distribution of states, conditional on receiving signal s. A voter is said to vote *informatively* if she votes for A when her signal is 1 and B when her signal is 0.

We need a last piece of notation before introducing our second theorem: for any state ω, let $\phi(A|\omega)$ denote the fraction of voters who prefer A to B, and $\phi(B|\omega)$ the fraction of voters who prefer B to A. We find:

Theorem 2 (A doubt is enough) *With private and common values, the two extreme states having strictly positive probability,* i.e. $F(0^+) > 0$, $F(1^-) < 1$, *is a sufficient condition for the* full information Condorcet winner *to be the only likely winner. That is:*
(i) A wins with a probability that converges to 1 when $n \to \infty$ in all states ω such that $\phi(A|\omega) > \phi(B|\omega)$ and
(ii) B wins with a probability that converges to 1 when $n \to \infty$ in all states ω' such that $\phi(A|\omega') < \phi(B|\omega')$.

Proof. See Appendix A3. ■

In this extended setup, equilibrium strategies are defined by two cutoffs: majority voters who have the most intense preferences in favour of A specialize in playing A, those who most intensely prefer B specialize in voting B, and those with the least intense preference double vote.

5.2 Approval Voting with Purely Partisan Voters

In Section 5.1, we showed that a doubt is enough to have FICE under Approval Voting. In this section, we show that this result extends to the case in which some voters have no such doubt, provided that they do not represent too large a fraction of the electorate. To introduce such purely partisan voters in the above setup, let the distribution of states be some $F(\omega)$ such that :

$$F(\omega) = 0,\ \forall\omega < \underline{\omega} \text{ and } F(\omega) = 1,\ \forall\omega > \bar{\omega}$$

with $0 < \underline{\omega} < \bar{\omega} < 1$. This implies that there is a fraction $\phi(B|\bar{\omega}) > 0$ of the electorate who prefers B to A with probability 1 and a fraction $\phi(A|\underline{\omega}) > 0$ of the electorate who prefers A to B with probability 1. These are purely partisan voters whose preference ordering is fixed no matter the state.

The following Proposition shows that FICE holds unless the fraction of purely partisan voters is too large:

Proposition 5 *When there are voters with no doubt, i.e. when the states $\omega < \underline{\omega}$ or $\omega > \bar{\omega}$ have zero-probability, Approval Voting produces a unique equilibrium which satisfies full information and coordination equivalence provided that:*

$$\max\left[\phi(A|\underline{\omega}), \phi(B|\bar{\omega})\right] < 1 - 2\sqrt{(1 - r(t_C))\, r(t_C)}.$$

Otherwise, there exist equilibria in which either A or B is the only likely winner in all states of nature.

Proof. See Appendix A3. ■

Theorems 1 and 2 showed that Approval Voting produces a unique equilibrium as long as the common value motive is sufficiently strong. This is the case when voters have a doubt about which alternative is best. Proposition 5 instead focuses on the case in which some majority voters do not have *any* doubt. The common value motive is thus inexistent for these voters. Myerson and Weber (1993) already showed that multiple equilibria coexist if all voters have purely private valued preferences. Here, Proposition 5 shows that FICE holds as long as the fraction of voters with *some* doubt is sufficiently large.

We want to argue that having some doubt is quite a natural assumption. Indeed, it seems obvious that there exists a least one candidate's deviant behavior that could make a voter change her mind. The examples in the introduction offer several concrete cases. FICE only requires that a candidate's wrongdoings are not perceived as entirely impossible and, as highlighted in the introduction, information relevant for the voters' perception of the candidates can come up, until the last minute.

Importantly, Proposition 5 does not imply that a large fraction of partisan voters is sufficient to exclude FICE in equilibrium. It only shows that there *also* exist equilibria in which FICE is not satisfied. Actually, there always exists an equilibrium that satisfies FICE in AV. This contrasts with other electoral systems like plurality, for which no strategy produces FICE if C is strong enough, and for which C may be the only likely winner in some equilibria.

5.3 Multinomial distribution

Until now, we assumed that the size of the population is random and follows a Poisson distribution. To show that our results do not hinge upon this assumption, we analyze here the polar case in which the size of the population is known and fixed. A traditional way to analyze large voting games with fixed population size is to consider a Multinomial distribution: the size of the population is fixed at n, and each voter is assigned a type $t \in \{t_A, t_B, t_C\}$ respectively with probabilities $r(t_A|\omega)$, $r(t_B|\omega)$ and $r(t_C|\omega)$. For a strategy function σ, the expected fraction of voters playing ψ in state ω is still defined by (6). The probability of a strategy profile $x = \{x(A), x(B), x(C), x(AB), x(AC), x(BC)\}$ in state ω is given by:

$$\Pr(x|\omega) = n! \times \Pi_{\psi} \frac{\tau(\psi|\omega)^{x(\psi)}}{x(\psi)!}.$$

Myerson (2000, Section 4) shows that, for n sufficiently large, pivot probabilities under such Multinomial distributions are simply a monotone transformation of their Poisson equivalent. In particular, the magnitude of a pivot probability in a Multinomial game is given by:

$$mag(piv_{PQ}|\omega, \text{Multinomial}) = \log(1 + mag(piv_{PQ}|\omega, \text{Poisson})).$$

It follows that the magnitude ratios behave exactly as in Poisson games. The only relevant difference between the multinomial and Poisson distributions is that, when expected vote share of an alternative P is exactly zero, the magnitude of the pivot probabilities involving P are $-\infty$ with the Multinomial distribution instead of -1 with the Poisson distribution. That is, a vote in favour of P could *never* be pivotal in the Multinomial game. Instead, pivot probabilities always are strictly positive in Poisson games. In other words, the assumption of a Poisson distribution acts as a tremble when the expected vote share of an alternative is zero: it operates "as if" the probability that types t_A (respectively t_B) vote for A (B) were bounded above zero. All our results extend directly to the case of a multinomial distribution if we introduce such a tremble, since the properties that we use in the proofs are the magnitude and offset ratios, and the latter are equivalent under the two distributions.[17]

Further, even in the absence of a tremble, corner solutions would not either be an equilibrium if voters had a lexicographic preference for approving of their preferred candidate: if a vote can never be pivotal, say in favour of A, t_A voters would vote AB instead of B. In that case as well, A and B must have a strictly positive vote share, which implies that all pivot probabilities must be strictly positive, and thus our results hold again.

6 Conclusion

We analyzed a three-alternative election in which a majority of the electorate is divided between two alternatives, A and B, and a minority supports a single alternative, C. In contrast to standard analyses of electoral systems, we introduced a common value component in the preferences of majority voters and showed that, when the common value component is sufficiently important and the size of the electorate sufficiently large, Approval Voting produces a unique equilibrium, in which the full information Condorcet winner is elected with a probability that approaches one. This result proves that the voters' sensitiveness to information influences the properties of Approval Voting in a fundamental way. We therefore argued that one must consider such sensitiveness to information when studying the properties of electoral systems. This has typically been overlooked by voting theory.

The reason why Approval Voting resists the problem of the divided majority when there is a common value component is twofold. First, Approval Voting allows voters to kill two birds with one ballot: they can vote for their most preferred alternative and lend support to their second choice when the minority candidate becomes a threat. Second, in the presence of a common value motive, voters want to allow their *a priori* second-best alternative to win the election in some states of nature. Hence, they develop an incentive to double-vote in order to balance the vote shares of the any alternative that *might* be best.

Arguably, the results rely on elaborate calculations from voters but the trade-offs and strategies that emerge are quite natural. First, the equilibrium strategy proves extremely intuitive: voters only need to understand that a multiple ballot is valuable whenever a potentially good candidate is too weak or when a disliked candidate gets too strong. Generalizing the setup to a continuum of types shows that the pattern of specialization that

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[17] Interestingly, the Poisson assumption also refines away bizarre equilibria in which *all* voters vote for the same candidate (A or B or C). With Multinomial distribution, all pivot probabilities are indeed exactly zero for such strategy profiles.

emerges is even more intuitive: voters who are closer to being indifferent between the two majority candidates double-vote, and those most in favor of either candidate single-vote.

Second, these trade-offs should also be robust to several extensions not considered in the paper. Think for instance of a world with more alternatives and states of nature. If there are k alternatives in the majority (and k associated states of nature), and l alternatives in the minority, the trade-off remains identical: as long as their primary objective is to fight one another, both majority-block and minority-block voters "multiple-vote" for their own alternatives. Within the majority, voters can multiple-vote to maintain the balance between their potentially good alternatives and make sure that each alternative wins in its associated state of nature. Indeed, our results show that, whenever an alternative trails behind, *all* majority-block voters want to support her with a multiple ballot. Hence, although the analysis would become much more cumbersome given the number of deviations to consider, the main insights remain. We can also think of a world in which C is not the worst alternative for majority-block voters: alternative A would still be the best in state a but would be the worst in state b, and vice versa for B. For that case, it is easy to prove the existence of an equilibrium that satisfies FICE: the strategy profile is exactly the same as in the initial setup.[18] This result shows that, despite different preferences, the full information Condorcet winner still ranks first and the alternative that can never be a full information Condorcet winner still ranks last. Unfortunately, in such a case, the proof of uniqueness becomes intractable because ballots including alternative C are no longer strictly dominated actions.

Finally, we considered a model in which alternatives/candidates are passive. A natural question for future research is to see how *candidates* behave when voters have common values. This analysis would be worth pursuing not only for Approval Voting but also for other electoral systems such as Plurality, Runoff and the Borda Count.

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[18] The proof is available upon request.

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Appendices

Appendices A1, A2 and A3 demonstrate the claims made in Sections 3, 4, and 5 respectively.

Appendix A1: Proofs for Section 3

Property 3 *(Myerson 2000, Theorem 2)) The probability that two actions, ψ and ψ' receive a number of votes that differs by a constant c ($c << n$) in state of the nature $\omega \in \{a, b\}$, is:*

$$\lim_{n\to\infty} \Pr\left(x(\psi) = x(\psi') + c|\omega\right) = \left(\frac{\tau(\psi|\omega)}{\tau(\psi'|\omega)}\right)^{c/2} \frac{\exp[-\left(\sqrt{\tau(\psi|\omega)} - \sqrt{\tau(\psi'|\omega)}\right)^2 n]}{2\sqrt{\pi n}\left(\tau(\psi|\omega)\tau(\psi'|\omega)\right)^{1/4}}.$$

Proof of Lemma 1. From Theorem 1 in Myerson (2000), the magnitude of the probability that alternatives A and C have (almost) the same number of votes is:

$$\begin{aligned} \lim_{n\to\infty} \tfrac{\log[\Pr(X(C)-X(A)\in\{0,1\}|\omega)]}{n} &= \max_x \sum_\psi \frac{x(\psi)}{n}\left(1 - \log\frac{x(\psi)}{n\tau(\psi|\omega)}\right) - 1 \\ & \qquad s.t.\ x(A) + x(AB) = x(C) + x(BC) \end{aligned} \tag{20}$$

If we denote $x(A) + x(AB) = x = x(C) + x(BC)$, $x(A) = \alpha x$, $x(AB) = (1-\alpha)x$, $x(C) = \beta x$ and $x(BC) = (1-\beta)x$, we find that this is maximized in:

$$\begin{aligned} \alpha^*_{AC} &= \frac{\tau(A|\omega)}{\tau(A|\omega) + \tau(AB|\omega)}, \\ x^*_{AC} &= n\sqrt{[\tau(C|\omega) + \tau(BC|\omega)][\tau(A|\omega) + \tau(AB|\omega)]}, \\ x(B)^*_{AC} &= n\tau(B|\omega), \\ \beta^*_{AC} &= \frac{\tau(C|\omega)}{\tau(C|\omega) + \tau(BC|\omega)}. \end{aligned} \tag{21}$$

Substituting for α^*_{AC}, x^*_{AC}, $x(B)^*_{AC}$ and β^*_{AC} in (20) thus yields:

$$\lim_{n\to\infty} \tfrac{\log[\Pr(X(C)-X(A)\in\{0,1\}|\omega)]}{n} = -\left(\sqrt{\tau(A|\omega) + \tau(AB|\omega)} - \sqrt{\tau(C|\omega) + \tau(BC|\omega)}\right)^2.$$

By analogy:

$$\begin{aligned} \lim_{n\to\infty} \tfrac{\log[\Pr(X(C)-X(B)\in\{0,1\}|\omega)]}{n} &= -\left(\sqrt{\tau(B|\omega) + \tau(AB|\omega)} - \sqrt{\tau(C|\omega) + \tau(AC|\omega)}\right)^2, \text{ and} \\ \lim_{n\to\infty} \tfrac{\log[\Pr(x(B)-x(A)\in\{0,1\}|\omega)]}{n} &= -\left(\sqrt{\tau(A|\omega) + \tau(AC|\omega)} - \sqrt{\tau(B|\omega) + \tau(BC|\omega)}\right)^2. \end{aligned}$$

Note the symmetry between $mag(PQ)$ and $mag(QP)$:

$$\lim_{n\to\infty} \tfrac{\log[\Pr(X(P)-X(Q)\in\{0,1\}|\omega)]}{n} = \lim_{n\to\infty} \tfrac{\log[\Pr(X(Q)-X(P)\in\{0,1\}|\omega)]}{n}.$$

■

Proof of Lemma 2. As explained in Section 3.1, the pivot probability between P and Q is the joint probability of two events. These two events can in fact be viewed as two constraints imposed on the number of votes to make a ballot pivotal: (i) Q is ahead of P by 0 or 1 vote and (ii) the 3^{rd} alternative, R, trails behind. To compute the magnitude of the different pivot probabilities, we use Theorem 1 in Myerson (2000) and impose these constraints. Applying this Theorem to compute the magnitude of the pivot probability between A and C gives:

$$\begin{aligned} mag(piv_{AC}|\omega) &= \max_{x} \sum_{\psi} x(\psi)\left(1-\log\frac{x(\psi)}{n\tau(\psi|\omega)}\right)-1 \\ & \quad s.t.\ x(A)+x(AB)=x(C)+x(BC) \text{ and } x(A)\geq x(B) \end{aligned} \tag{22}$$

If we abstract from the constraint $x(C)+x(BC)\geq x(B)+x(AB)$, or if this constraint is not binding, (22) is maximized for α^*_{AC}, x^*_{AC}, $x(B)^*_{AC}$ and β^*_{AC} as defined in (21). Substituting for α^*_{AC}, x^*_{AC}, $x(B)^*_{AC}$ and β^*_{AC} in (22) yields:

$$mag(piv^*_{AC}|\omega)=\lim_{n\to\infty}\frac{\log[\Pr(|X(C)-X(A)|\leq 1|\omega)]}{n}=-\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)+\tau(BC|\omega)}\right)^2.$$

We refer to this as the *unrestricted* magnitude (denoted by *).

If the constraint is binding, i.e. if $\alpha^*_{AC}x^*_{AC}\leq x(B)^*_{AC}$, the joint probability also depends on another event that has a strictly negative magnitude. Taking this constraint into account implies:

$$mag(piv_{AC}|\omega)\leq mag(piv^*_{AC}|\omega)=-\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)+\tau(BC|\omega)}\right)^2.$$

By analogy, it is immediate to check that:

$$\begin{aligned} mag(piv_{BC}|\omega) &\leq mag(piv^*_{BC}|\omega)=-\left(\sqrt{\tau(B|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)+\tau(AC|\omega)}\right)^2, \\ \text{and } mag(piv_{AB}|\omega) &\leq mag(piv^*_{AB}|\omega)=-\left(\sqrt{\tau(A|\omega)+\tau(AC|\omega)}-\sqrt{\tau(B|\omega)+\tau(BC|\omega)}\right)^2. \end{aligned}$$

Now, note that the three events piv_{AB}, piv_{AC} and piv_{BC} are identical if their respective constraints are binding. Indeed, whatever the event, a binding constraint implies: $x(A)+x(AB)+x(AC)=x(C)+x(AC)+x(BC)=x(B)+x(AB)+x(BC)$. We refer to the magnitude of this binding events as the *restricted* magnitudes (denoted by **):

$$mag(piv^{**}_{AC}|\omega)=mag(piv^{**}_{BC}|\omega)=mag(piv^{**}_{AB}|\omega),$$

which, by definition, are smaller than the lowest unrestricted magnitude:

$$mag(piv^{**}_{AC}|\omega)\leq \min_{P,Q\in\{A,B,C\}} mag\left(piv^*_{PQ}|\omega\right).$$

Having observed this, we are we are now in a position to prove that, if the expected ranking is $A>C>B$ in state ω, then:

$$\begin{aligned} mag(piv_{AC}|\omega) &= \mu(A,B) \text{ and} \\ mag(piv_{BC}|\omega) &< \mu(B,C). \end{aligned}$$

First, we need to prove that $mag\left(piv_{AC}|\omega\right)$ is unrestricted. This is true if $x\left(A\right)+x\left(AB\right)=x\left(C\right)+x\left(BC\right)$ implies $x\left(A\right)+x\left(AB\right)>x\left(B\right)+x\left(AB\right)$ at the optimum, that is:

$$\alpha_{AC}^{**}x_{AC}^{**}>x\left(B\right)_{AC}^{**}.$$

Using (21) and performing some manipulations, we see that the latter inequality holds iff:

$$\sqrt{\frac{\tau\left(C|\omega\right)+\tau\left(BC|\omega\right)}{\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)}}>\frac{\tau\left(B|\omega\right)}{\tau\left(A|\omega\right)}, \tag{23}$$

in which both sides are smaller than one. Hence: $\frac{\tau(B|\omega)}{\tau(A|\omega)}\leq\frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}\leq\sqrt{\frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}}$, and by the assumed expected ranking $A>C>B$ in state ω, the last member of this inequality is smaller than $\sqrt{\frac{\tau(C)+\tau(BC|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}}$, which proves that $mag(piv_{AC}|\omega)$ is always unrestricted. Hence $mag(piv_{AC}|\omega)=\mu\left(A,B\right)$.

Second, we need to prove that $mag\left(piv_{BC}|\omega\right)$ is restricted. This is true if $x\left(B\right)+x\left(AB\right)=x\left(C\right)+x\left(AC\right)$ implies $x\left(A\right)+x\left(AB\right)>x\left(B\right)+x\left(AB\right)$ at the optimum, that is:

$$\alpha_{BC}^{**}x_{BC}^{**}<x\left(A\right)_{BC}^{**} \tag{24}$$

where

$$\begin{aligned}\alpha_{BC}^{**} &= \frac{\tau\left(B|\omega\right)}{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)},\\ x_{BC}^{**} &= n\sqrt{[\tau\left(C|\omega\right)+\tau\left(AC|\omega\right)][\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)]},\\ x\left(A\right)_{BC}^{**} &= n\tau(A|\omega).\end{aligned}$$

(the derivation of these critical values α_{BC}^{**}, x_{BC}^{**}, and $x(A)_{BC}^{**}$ is identical to that of α_{AC}^{**}, x_{AC}^{**}, and $x\left(B\right)_{AC}^{**}$ in (21)). To show that (24) holds, we proceed as with (23) and show that:

$$\sqrt{\frac{\tau\left(C|\omega\right)+\tau\left(AC|\omega\right)}{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)}}<\frac{\tau\left(A|\omega\right)}{\tau\left(B|\omega\right)},$$

in which both fractions are larger than one. This implies: $\frac{\tau(A|\omega)}{\tau(B|\omega)}\geq\frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}\geq\sqrt{\frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}}$ and, by the assumed expected ranking $A>C>B$ in state ω, the last member of this inequality is always larger than $\sqrt{\frac{\tau(C|\omega)+\tau(BC|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}}$, which proves that $mag(piv_{BC}|\omega)$ is always restricted and completes the proof.

The proof is identical for all the other possible rankings: $C>B>A$, $C>A>B$ and $B>C>A$, which proves the Lemma. ■

Proof of Proposition 1. $\sigma^{IT}\left(t_A\right)=\sigma^{IT}\left(t_B\right)$ follows immediately from (6). Next, from (8) we have that $\Pr\left(piv_{PQ}|a\right)=\Pr\left(piv_{PQ}|b\right)$ for all $P,Q=A,B,C$ if the distribution of votes for each alternative is the same in both states of nature. Introducing this in (10)-(12) directly shows that $G\left(AB|t_A\right)>G\left(B|t_A\right)$ and $G\left(AB|t_B\right)>G\left(A|t_B\right)$ for any strategy function $\sigma^{IT}\left(t_A\right)=\sigma^{IT}\left(t_B\right)$. This in turn implies that $\sigma\left(A|t_B\right)=0=\sigma\left(B|t_A\right)$, which by (6) implies that $\sigma\left(AB|t_A\right)=1=$

$\sigma(AB|t_B)$ is the only strategy profile in undominated strategies that produces an informational trap. ■

Proof of Proposition 2. By Proposition 1, the only candidate strategy function for an informational trap is $\sigma(AB|t_A) = 1 = \sigma(AB|t_B)$, which would imply that $X(A) = X(B)$ for any realized number of voters. By (3), we have: $\mathsf{E}(X(A)) > \mathsf{E}(X(C))$. By the law of large numbers and the magnitude theorem (see Properties 1 and 2), we have:

$$\begin{array}{ll} (i) & \Pr(piv_{AB}|\omega, n) < 1/2 \text{ and } \lim_{n\to\infty} \Pr(piv_{AB}|\omega, n) = 1/2, \\ (ii) & \Pr(piv_{PC}|\omega, n) > 0 \text{ and } \lim_{n\to\infty} \Pr(piv_{PC}|\omega) = 0 \text{ for } P \in \{A, B\}. \end{array}$$

Since these pivot probabilities are monotonic in n, there must exist some $\bar{n}$ such that

$$\max\left[G(A|t, n \geq \bar{n}), G(B|t, n \geq \bar{n})\right] > G(AB|t, n \geq \bar{n}).$$

■

Appendix A2: Proofs for Section 4

Lemma 5

$$G(A|t) \geq G(AB|t) \iff \tfrac{q(b|t)}{q(a|t)} \leq \frac{1}{M_1} \equiv \tfrac{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)} \tag{25}$$

$$G(B|t) \geq G(AB|t) \iff \tfrac{q(a|t)}{q(b|t)} \leq M_2 \equiv \tfrac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)} \tag{26}$$

Proof. Immediate from (10), (11) and (12). ■

Proof of Proposition 4. A necessary condition for A and B to be played with positive probability in equilibrium is that, for some $t \in \{t_A, t_B\}$:

$$G(A|t) = G(B|t) \geq G(AB|t), \tag{27}$$

and, from Lemma 5 (in this Appendix), $G(A|t), G(B|t) \geq G(AB|t)$ require $\Pr(piv_{AB}|a) > \Pr(piv_{BC}|a)$ and $\Pr(piv_{BA}|b) > \Pr(piv_{AC}|b)$.

Using (10) and (11), a necessary condition for $G(A|t) = G(B|t)$ is:

$$\frac{q(a|t)}{q(b|t)} = \frac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b) + \Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a) + \Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)}. \tag{28}$$

Now, we prove that (27) can never hold: using Lemma 5 (in this Appendix), we identify a lower bound for M_1 and an upper bound for M_2. Then, we show that this lower bound for M_1 is strictly larger than the upper bound for M_2, whereas condition (27) requires:

$$M_1 \leq M_2, \tag{29}$$

hence the contradiction.

$M_1 = \frac{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)}$ is strictly increasing in $\Pr(piv_{BC}|a)$ and $\Pr(piv_{BC}|b)$. A lower bound to M_1 is thus found by setting these two pivot probabilities equal to 0. Similarly, an upper bound to M_2 is found by setting $\Pr(piv_{AC}|a)$ and $\Pr(piv_{AC}|b)$ equal to zero. This establishes that:

$$\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)} < M_1 \text{ and } M_2 < \frac{\Pr(piv_{BA}|b)}{\Pr(piv_{BA}|a)}, \tag{30}$$

and hence that a necessary condition for (29) is that:

$$\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{BA}|b)}\frac{\Pr(piv_{BA}|a)}{\Pr(piv_{AB}|a)} < 1.$$

Using Property 3 (in Appendix A1), the left-hand side of this expression is equal to:

$$\sqrt{\frac{\tau(A|a)}{\tau(A|b)}\frac{\tau(B|b)}{\tau(B|a)}},$$

which cannot be smaller than 1. Indeed, by (28), types t_A must vote for A with a higher probability than types t_B, since $\frac{q(a|t_A)}{q(b|t_A)} > \frac{q(a|t_B)}{q(b|t_B)}$. Hence, in equilibrium:

$$\frac{\tau(A|a)}{\tau(A|b)} \geq 1 \text{ and } \frac{\tau(B|b)}{\tau(B|a)} \geq 1. \tag{31}$$

It follows that $G(A|t) = G(B|t)$ implies $G(AB|t) > G(A|t)$, and therefore that a strict mixture between A and B is a strictly dominated strategy: $\sigma(A|t) > 0$ implies $\sigma(B|t) = 0$ and conversely. ■

Proof of Proposition 3. We have to prove that $\sigma(A|t_A)$ and $\sigma(B|t_B)$ are strictly positive in equilibrium. To this end, we show that:

$$\sigma(B|t_B) > 0 \text{ and } \sigma(A|t_A) = 0 \tag{32}$$

leads to a contradiction. Indeed, (32) implies $\tau(A|\omega) = 0$ in both states. Hence, by Property 1:

$$\mu(A,B) = -\tau(B|\omega).$$

By (31), we have: $\tau(B|a) < \tau(B|b)$. This implies that $\lim_{n\to\infty} \Pr(piv_{BA}|b)/\Pr(piv_{BA}|a) = 0$ and therefore that $\lim_{n\to\infty} M_2 \leq 0$ in Lemma 5. Instead, $\sigma(B|t_B) > 0$ imposes that M_2 be strictly positive. This shows that $\sigma(A|t_A) = 0$ contradicts the possibility that $\sigma(B|t_B) > 0$. By symmetry, we cannot either have: $\sigma(A|t_A) > 0$ and $\sigma(B|t_B) = 0$.

Together with Proposition 2 and Lemma 4, this proves that, in equilibrium, we must have $\sigma(A|t_A) > 0$ and $\sigma(B|t_B) > 0$. ■

Proof of Proposition 4.

To prove that there is an unique equilibrium, we proceed in two steps. First, we show that $\sigma(A|t_A) =$

$\rho^{*}\sigma\left(B|t_{B}\right)$ is the unique best response of types t_A given the strategy of types t_B. Second, we prove that there is a unique equilibrium strategy $\sigma^{*}\left(B|t_{B}\right)$.

From (16) and (18), we must have in equilibrium:

$$\begin{aligned} mag\left(piv_{AB}|a\right) = mag\left(piv_{AB}|b\right) \geq \quad & \max\{mag\left(piv_{BC}|a\right), mag\left(piv_{BC}|b\right), \\ & mag\left(piv_{AC}|a\right), mag\left(piv_{AC}|b\right)\}. \end{aligned} \tag{33}$$

We can check that types t_A never want to deviate from $\sigma\left(A|t_{A}\right) = \rho^{*}\sigma\left(B|t_{B}\right)$: for any $\sigma\left(A|t_{A}\right) < \rho^{*}\sigma\left(B|t_{B}\right)$, we have $\sigma\left(AB|t_{A}\right) > 1 - \rho^{*}\sigma\left(B|t_{B}\right)$. This implies that the expected share of alternative B increases in both states and hence that: $mag\left(piv_{AB}|a\right)$ increases above $mag\left(piv_{AB}|b\right)$, whereas $mag\left(piv_{BC}|a\right)$ and $mag\left(piv_{BC}|b\right)$ decrease.

Using Lemma 5 (in Appendix A1) and (33), this implies:

$$\frac{q\left(b|t_{A}\right)}{q\left(a|t_{A}\right)} < \lim_{n\to\infty}\frac{1}{M_1} \equiv \frac{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)} = \infty,$$

and hence: $G(A|t_A) > G\left(AB|t_{A}\right)$. Therefore, $\sigma\left(A|t_{A}\right) < \rho^{*}\sigma\left(B|t_{B}\right)$ cannot be true in equilibrium.

For any $\rho^{*}\sigma\left(B|t_{B}\right) < 1$, we also have to check that $\sigma\left(A|t_{A}\right) > \rho^{*}\sigma\left(B|t_{B}\right)$ cannot be an equilibrium either. Following the same procedure as above, one can check that $\sigma\left(A|t_{A}\right) > \rho^{*}\sigma\left(B|t_{B}\right)$ implies:

$$\frac{q\left(b|t_{A}\right)}{q\left(a|t_{A}\right)} > \lim_{n\to\infty}\frac{1}{M_1} \equiv \frac{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)} \leq 0,$$

which in turn implies $G(A|t) < G\left(AB|t\right)$. Hence, $\sigma\left(A|t_{A}\right) > \rho^{*}\sigma\left(B|t_{B}\right)$ cannot be true in equilibrium. Therefore, when (33) holds, $\sigma^{*}\left(A|t_{A}\right) = \rho^{*}\sigma\left(B|t_{B}\right)$ is the unique best response of types t_A to $\sigma\left(B|t_{B}\right)$.

It remains to prove that there is a unique equilibrium strategy $\sigma^{*}\left(B|t_{B}\right)$, which will always imply (33). Two cases must be considered:

Case 1: $G(B|t_B) - G(AB|t_B) \geq 0$ in $\sigma\left(B|t_{B}\right) = 1$, $\sigma\left(A|t_{A}\right) = \rho^{*}$.

In that case, $\sigma\left(B|t_{B}\right) = 1$ is the only possible best response for types t_B. Indeed, $\sigma\left(B|t_{B}\right) < 1$ would imply $\sigma\left(AB|t_{B}\right) > 0$. This induces an increase in the expected vote share of alternative A in both states of nature and hence that: $mag\left(piv_{BA}|b\right)$ increases above $mag\left(piv_{BA}|a\right)$, whereas $mag\left(piv_{AC}|a\right)$ and $mag\left(piv_{AC}|b\right)$ decrease. Using Lemma 5 and (33), this implies:

$$\frac{q\left(a|t_{B}\right)}{q\left(b|t_{B}\right)} < \lim_{n\to\infty} M_2 \equiv \frac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)} = \infty,$$

and hence $G(B|t_B) > G\left(AB|t_{B}\right)$. Therefore, $\sigma\left(B|t_{B}\right) = 1$ is the unique best response to $\sigma\left(A|t_{A}\right) = \rho^{*}$.

It remains to show that types t_B would deviate from any $\{\sigma\left(A|t_{A}\right), \sigma\left(B|t_{B}\right)\} = \{\rho^{*}\sigma, \sigma\}$ if $\sigma < 1$. To this end, we need to show that

$$\lim_{n\to\infty}\frac{G(B|t_B) - G(AB|t_B)}{\Pr(piv_{AB}|a)} = q(b|t_B)\frac{\Pr(piv_{BA}|b)}{\Pr(piv_{AB}|a)} - q(a|t_B)\frac{\Pr(piv_{BA}|a)}{\Pr(piv_{AB}|a)} > 0, \tag{34}$$

for any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\sigma, \sigma\}$, $\sigma < 1$.

The strategy of the types t_A implies:

$$\begin{aligned}\lim_{n\to\infty}\frac{G(A|t_A)-G(AB|t_A)}{\Pr(piv_{AB}|a)} &= q(a|t_A)-q(b|t_A)\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)}=0\\ &\Longrightarrow \frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)}=\frac{q(a|t_A)}{q(b|t_A)}.\end{aligned}$$

By Myerson's offset theorem (Property 3 in Appendix A1): $\Pr(piv_{BA}|\omega) = \Pr(piv_{AB}|\omega)\sqrt{\frac{\tau(A|\omega)}{\tau(B|\omega)}}$. Hence, (34) can be rewritten as:

$$\frac{q(b|t_B)}{q(a|t_B)}\frac{q(a|t_A)}{q(b|t_A)} > \sqrt{\frac{\tau(A|a)\,\tau(B|b)}{\tau(B|a)\,\tau(A|b)}}.$$

By (4), the left-hand side of this inequality is equal to: $\frac{\tau(A|a)\tau(B|b)}{\tau(B|a)\tau(A|b)} > 1$, which proves that (34) holds.

Case 2: $G(B|t_B) - G(AB|t_B) < 0$ in $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \rho^*$.

In this case, there must exist a $\bar{\sigma} \in (0,1)$ such that, for $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\bar{\sigma}, \bar{\sigma}\}$, we have: $G(B|t_B) - G(AB|t_B) = 0$. Indeed, by Proposition 2, $G(B|t_B) - G(AB|t_B) > 0$ for $\sigma(A|t_A) = 0 = \sigma(B|t_B)$. The existence of $\bar{\sigma}$ immediately follows from the continuity of the G function.

This value of $\bar{\sigma}$ is unique and such that:

$$\begin{aligned} mag(piv_{AB}|a) = mag(piv_{AB}|b) = \ & \max\{mag(piv_{BC}|a), mag(piv_{BC}|b),\\ & mag(piv_{AC}|a), mag(piv_{AC}|b)\}.\end{aligned} \tag{35}$$

Indeed, any $\sigma < \bar{\sigma}$ implies that the total expected vote shares of alternatives A and B increase. Since (35) implies that C is third in both states, the magnitudes $mag(piv_{PC}|\omega)$ must decrease, for any $P \in \{A, B\}$ and $\omega \in \{a, b\}$. In contrast, the magnitudes $mag(piv_{AB}|\omega)$ must increase, since:

$$\begin{aligned} mag(piv_{AB}|a) = mag(piv_{AB}|b) &= -\left(\sqrt{r(t_A|a)\cdot\rho^*\sigma}-\sqrt{r(t_B|a)\cdot\sigma}\right)^2\\ &= -\left(\sqrt{r(t_A|a)\cdot\rho^*}-\sqrt{r(t_B|a)}\right)^2\sigma\end{aligned}$$

is strictly increasing in σ. Hence (33) holds with a strict inequality for any $\sigma < \bar{\sigma}$. This implies that (34) holds, and hence that $G(B|t_B) - G(AB|t_B) > 0$ for any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\sigma, \sigma\}$, $\sigma < \bar{\sigma}$.

Similarly, one can check that (33) is violated for any $\sigma > \bar{\sigma}$ which implies $G(B|t_B) - G(AB|t_B) < 0$ for any $\sigma(A|t_A) = \rho^*\sigma$ and $\sigma(B|t_B) = \sigma$, with $\sigma > \bar{\sigma}$. This proves that (35) must hold in $\sigma(A|t_A) = \rho^*\bar{\sigma}$ and $\sigma(B|t_B) = \bar{\sigma}$, and that the solution to $\bar{\sigma}$ is unique. ■

Appendix A3: Proof of Section 5

Proof of Theorem 2. Like in the simple setup, majority voters choose one out of three actions: vote A, vote B, or vote AB. Denote $\Delta U_i(PQ,\omega) \equiv U_i(P,\omega) - U_i(Q,\omega),\ P,Q \in \{A,B,C\}$. The payoff associated with each action is:

$$\begin{aligned}
G_i(A|s) &= \int_0^1 q(\omega|s)\left[\Pr(piv_{AB}|\omega)\ \ \Delta U_i(AB|\omega) + \Pr(piv_{AC}|\omega)\ \ \Delta U_i(AC|\omega)\right] d\omega \\
G_i(B|s) &= \int_0^1 q(\omega|s)\left[-\Pr(piv_{BA}|\omega)\ \ \Delta U_i(AB|\omega) + \Pr(piv_{BC}|\omega)\ \ \Delta U_i(BC|\omega)\right] d\omega \\
G_i(AB|s) &= \int_0^1 q(\omega|s)\left[\Pr(piv_{AC}|\omega)\ \ \Delta U_i(AC|\omega) + \Pr(piv_{BC}|\omega)\ \ \Delta U_i(BC|\omega)\right] d\omega.
\end{aligned}$$

We know that pivot probabilities are determined by the action profile of the voters. Given an action profile σ, let mag_1 denote the largest magnitude:

$$mag_1 \equiv \max_{\omega}\left[\max\left\{mag(piv_{AB}|\omega), mag(piv_{AC}|\omega), mag(piv_{BC}|\omega)\right\}\right],$$

and let ω_1 be the arg max of that expression. Similarly, the second largest magnitude is denoted mag_2 and its arg max is ω_2.

Now, we show by contradiction that the largest magnitude must be the one between A and B: imagine that $mag_1 = mag(piv_{AC}|\omega_1)$. Then, it is straightforward to check that $G_i(AB|s) > G_i(B|s)$, $\forall i, s$. Thus, all majority voters must either vote A or AB.

Since there cannot be an informational trap, the excess vote share of A over B must thus be increasing in ω, which implies: $mag_2 = mag(piv_{AB}|\omega_2 = 0)$. But in that case, no majority voter wants to be pivotal against B ($U_i(A,0) - U_i(B,0) < 0,\ \forall i$): all would vote AB, which leads to a contradiction. By symmetry, the same holds for $mag_1 = mag(piv_{BC}|\omega_1)$. Thus, mag_1 must be $mag(piv_{AB}|\omega_1)$.

Now, we need to show that (a) ω_1 is an interior state, and (b) that $\phi(A|\omega_1) = \phi(B|\omega_1)$: (a) if $\omega_1 \in \{0,1\}$, then all majority voters would vote "en bloc" either A or B, which leads to an informational trap; a contradiction. If instead $\omega_1 \in (0,1)$, then some voters i prefer A to B in state ω_1, and either vote A or AB (and conversely for those who prefer B to A).

(b) we show that $\phi(A|\omega_1) = \phi(B|\omega_1)$: if $\phi(A|\omega_1) > \phi(B|\omega_1)$, then the expected number of voters who play A must be larger than the expected number of voters who play B (remember that $mag_1 = mag(piv_{AB}|\omega_1)$ thus $\tau(A|\omega_1) = \phi(A|\omega_1)$ and $\tau(B|\omega_1) = \phi(B|\omega_1)$). But there must also exist a state $\omega < \omega_1$ for which $\phi(A|\omega_1) > \phi(A|\omega) > \phi(B|\omega) > \phi(B|\omega_1)$, which implies $\tau(A|\omega_1) > \tau(A|\omega) > \tau(B|\omega) \geq \tau(B|\omega_1)$, and therefore $mag(piv_{AB}|\omega) > mag(piv_{AB}|\omega_1)$. This contradicts the definition of ω_1. Since the same reasoning can be applied to $\phi(A|\omega_1) < \phi(B|\omega_1)$, it must be that $\tau(A|\omega_1) = \tau(B|\omega_1)$, which requires $\phi(A|\omega_1) = \phi(B|\omega_1)$ and implies $mag(\omega_1) = 0$. Finally, this also implies that $\tau(A|\omega) \gtrless \tau(B|\omega)$ iff $\phi(A|\omega) \gtrless \phi(B|\omega)$. Note that only the voters close to being indifferent between A and B in state ω_1 vote informatively. ■

Proof of Proposition 5. We show that, for $\phi(B|\bar{\omega})$ large enough, there exists an equilibrium in which B always wins in Approval Voting. We are looking for a necessary condition under which, for n large enough, there exists an equilibrium such that the fraction of voters who vote A is zero: $\tau(A|\omega) = 0$, whereas the fraction $\phi(B|\bar{\omega})$ of types i s.t. $\Delta U_i\left(AB|\bar{\omega}\right) < 0$ vote for B. The other majority voters play AB. This strategy function implies that B has a strictly larger vote share than A in all states of nature.

We show below that such an equilibrium is sustained by the gains:

$$\begin{aligned} G_i(AB|s) &\geq \max\left\{G_i(A|s), G_i(B|s)\right\}, \forall i, s \text{ s.t. } \Delta U_i\left(AB|\bar{\omega}\right) > 0 \text{ and} && (36)\\ G_i(B|s) &\geq G_i(AB|s)\ \forall i, s \text{ s.t. } \Delta U_i\left(AB|\bar{\omega}\right) < 0. && (37)\end{aligned}$$

Consider the following gain differentials:

$$G_i\left(B|s\right) - G_i\left(AB|s\right) = \int_0^1 q\left(\omega|s\right)\left[\Pr\left(piv_{BA}|\omega\right)\ \Delta U_i\left(BA|\omega\right) - \Pr\left(piv_{AC}|\omega\right)\ \Delta U_i\left(AC|\omega\right)\right] d\omega,$$

$$G_i\left(A|s\right) - G_i\left(AB|s\right) = \int_0^1 q\left(\omega|s\right)\left[\Pr\left(piv_{AB}|\omega\right)\ \Delta U_i\left(AB|\omega\right) - \Pr\left(piv_{BC}|\omega\right)\ \Delta U_i\left(BC|\omega\right)\right] d\omega.$$

From Feddersen and Pesendorfer (1997), we know that the fraction of voters who vote informatively goes to zero as $n \to \infty$. Yet, the number of such voters is increasing in n. Thus, the excess vote share of B over A must be decreasing in ω for any n. For the strategy function considered above, the probability of state $\bar{\omega}$ conditional on being pivotal between A and B therefore increases towards 1 as n goes to infinity. Hence, the above differentials converge to:

$$G_i\left(B|s\right) - G_i\left(AB|s\right) \underset{n\to\infty}{\to} \Pr\left(piv_{BA}|\bar{\omega}\right)\ \Delta U_i\left(BA|\bar{\omega}\right) - \int_0^1 q\left(\omega|s\right)\Pr\left(piv_{AC}|\omega\right)\ \Delta U_i\left(AC|\omega\right)\ d\omega, \tag{38}$$

$$G_i\left(A|s\right) - G_i\left(AB|s\right) \underset{n\to\infty}{\to} \Pr\left(piv_{AB}|\bar{\omega}\right)\ \Delta U_i\left(AB|\bar{\omega}\right) - \int_0^1 q\left(\omega|s\right)\Pr\left(piv_{BC}|\omega\right)\ \Delta U_i\left(BC|\omega\right)\ d\omega. \tag{39}$$

Since A and C rank second and third in both states of nature, the magnitude of $\Pr(piv_{AC}|\omega)$ is restricted while the magnitude of $\Pr(piv_{BA}|\omega)$ is unrestricted for all ω. This implies that, for all ω, $\Pr\left(piv_{BA}|\bar{\omega}\right)/\Pr\left(piv_{AC}|\omega\right) \to \infty$ as $n \to \infty$ (see Lemma 2 and Property 2) . The differential (38) is thus only positive if $\Delta U_i\left(BA|\bar{\omega}\right) > 0$, which is the case for the partisan types only, who represent a fraction $\phi(B|\bar{\omega})$ of the electorate. For the other voters, (39) is non-positive only if $\Pr\left(piv_{AB}|\bar{\omega}\right)/\Pr\left(piv_{BC}|\omega\right)$ becomes sufficiently small as $n \to \infty$. Noting that the vote shares of B and C are constant for all states of nature, this in turn requires that $mag\left(piv_{AB}|\bar{\omega}\right) \leq mag\left(piv_{BC}|\bar{\omega}\right)$. That is:

$$mag(piv_{AB}|\bar{\omega}) = -\phi(B|\bar{\omega}) \leq -\left(\sqrt{1-r(t_C)} - \sqrt{r(t_C)}\right)^2 = mag(piv_{BC}|\bar{\omega}),$$

which boils down to $\phi(B|\bar{\omega}) \geq 1 - 2\sqrt{\left(1-r(t_C)\right)r(t_C)}$. The proof for $\phi(A|\underline{\omega})$ is similar. ■