Victory margins and the paradox of voting

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Abstract

This paper addresses a simple question: why do people vote? Though simple, this question remains unanswered despite the considerable attention it has received. In this paper, I show that purely rational/instrumental factors explain a large fraction of turnout variations provided that the effect of the margin of victory on implemented policy is considered. I extend Myerson's models of elections based on Poisson games, and show that, when platforms are responsive to vote shares, the predictions of the model become consistent with several stylized facts, including the secular fall in turnout rates in the U.S.

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1 Introduction

> *"To measure the political preferences of legislators by their votes at year 1 and, then, to use those very votes to explain their behavior at year 1 is to argue that legislators vote the way they do because they vote the way they do."*
> (Epstein and Mershon, 1996 pp261-262)

Representative democracy includes different ingredients: elections, candidates, and voters. Electoral rules then determine how voters' ballots translate into representation, and how candidates compete, how budgets will be spent, and so on. Electoral rules thus determine the personal value of voting, which in turn influences turnout rates.

This paper studies this last problem: why do people vote? Explaining turnout is difficult because each single vote has a negligible weight on the outcome of the election and, if this weight is so low, why not always abstain? Put differently, the mere fact that turnout can be large (105 million voters turned out at the U.S. presidential elections, in 2000) appears paradoxical. Incurring the costs of voting would seem to indicate irrationality.

I show, however, that rational behaviour can be consistent with large turnouts. For this result to hold, I consider (1) the uncertainty that voters face at the time of the election and (2) the fact that the elected party generally behaves differently if it wins by

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a large or a small margin. In other words, I show that if, following the results of Razin (2000), we accept that the platform implemented after the election not only depends on which party is elected, but also on the apparent *mandate* of the elected party or candidate, then the value of a vote is higher than traditionally perceived.

This paper is part of a vast literature that tries to reconcile (possibly bounded) rationality with the large turnouts that are generally observed. This literature provides little agreement, except on one point: turnout does not seem to be motivated by instrumental arguments. "Instrumental voting" means that a voter goes to the polls in order to affect the outcome of the election. That is, voters use elections as an instrument to influence policy. Riker and Ordeshook (1968) were the first to identify the exact weight of a vote. As they show, a vote is only effective if it is *pivotal*, i.e. if it changes the outcome of the election. Therefore, the instrumental value of a vote is proportional to the *probability* of being pivotal. In a large population, this probability is extremely small and cannot match any positive cost of voting.

For this reason, other rationales for the turnout/abstention decision have been explored. As alternatives to the rational-instrumental voter hypothesis, one could imagine that voters are actually not rational (e.g. Ferejohn and Fiorina (1974) assume that voters do not even consider this probability of being pivotal, and behave as if it were equal to one), or at least not instrumental. Under the latter assumption, variations in turnout are caused by a mechanism that directly affects the *cost* of voting. For instance, Aldrich (1993) argues that parties voluntarily reduce the cost of voting if the election is very important to them; Harbaugh (1996) presents a model where voters turn out because of peer pressure; Feddersen and Pesendorfer (1996) argue that voters may rationally abstain when they are ill-informed about the relative merits of the candidates.[1]

The starting point of this paper is that these debates may have put too much emphasis on the voters' motivations, and not enough on other aspects of elections. I propose that the rational-instrumental approach still has something to teach us. However, additional linkages between voting behaviour and the policy implemented after the election must be considered. A crucial, yet overlooked, aspect to explore is the effect of the margin of victory on implemented policy. One step in that direction relates to the way the margin of victory influences parties and hence for which party voters cast their ballot (Razin, 2000; Castanheira, 2002). Another step relates to the effects of this margin on the voters' decision to turn out. This is the purpose of this paper.[2]

Another contribution of the paper is to extend Myerson's (1997, 2000) models of elections based on *Poisson games*, and solve for the equilibrium turnout when the effects of the *mandate* are taken into account. The tractability of Myerson's models allow a simple solution to the apparently complex interactions introduced by the effects of mandates.

The paper proceeds as follows. Section 2 introduces the model and the preferences of the voters. Section 3 derives the equilibrium level of turnout under the same assumptions as in the rest of the literature, and shows how unsatisfactory these results are. Section 4 relaxes the assumption that voters exactly know the preferences of the

[1] For a more complete survey, see for instance Mueller (1989) or Aldrich (1993, 1997).

[2] Note that a second prediction of the rational/instrumental approach is that of *rational ignorance*: a voter who knows that he or she has little weight on the election outcome would not incur the costs of acquiring information. Conversely, when this weight increases, voters want to acquire more information, and newspapers should provide more information about closely contested constituencies. That aspect of the rational-instrumental hypothesis is tested –and verified– by Larcinese (2002).

whole electorate. In Section 5, Stigler's (1972) and Razin's (2000) results are used to study the case in which the "mandate" given to the politician matters in determining the implemented policy.

In Section 6, these results are applied to numerical examples. When I calibrate the results of Section 5 to U.S. or international data, a substantial part of the fall in turnout since 1960 is explained by the model. Section 7 concludes. Some of the computations are in the appendix, which also presents a short survey of various empirical regularities highlighted in the literature.

2 The model

I consider a game in which voters must decide whether to turn out or to abstain in an upcoming election, and I shall be looking for Bayesian Nash Equilibria of this game. The reference model is that of Riker and Ordeshook (1968) – RO for short. In that model, the utility of the voters depends both on the platform that is implemented and on their decision to turn out or to abstain. Assuming that the policy space is unidimensional, voter i's preferred platform can be represented by a real number θ_i. Similarly, the implemented policy is defined as another real number P. I define the utility of voter i as:

$$U(\theta_i, \mathsf{P}, v_i) = -(\theta_i - \mathsf{P})^2 + c(v_i), \tag{1}$$

in which the first member on the right-hand side of (1) denotes the utility derived from the implemented policy P and $c(v_i)$ represents the net cost of voting.

Assume that there are two candidates, with policy platforms A and B. Denoting abstention by $\varnothing$, voters can choose one in three actions: vote for A (action A), vote for B (action B) or abstain (action $\varnothing$):

$$v_i \in \{A, B, \varnothing\}.$$

Introducing the cost of turning out is made by posing:

$$\begin{aligned} c(A) &= c(B) = -c_i, \\ c(\varnothing) &= 0. \end{aligned}$$

Thus, if $c_i > 0$, voter i has a positive cost of voting, and the voter instead has a positive cost of abstaining if $c_i < 0$ (c_i is thus the gross cost of voting *net of* any sense of duty).[3]

From these definitions, one can immediately derive the value of a ballot. By turning out, there is some probability that the voter is pivotal; *i.e.* that his or her ballot affects the outcome of the election. Hence, the value of the ballot depends on this *pivot probability*:

$$\begin{aligned} \mathsf{E}[U(\theta_i, \mathsf{P}, A) - U(\theta_i, \mathsf{P}, \varnothing)] &= \mathsf{E}_{\mathsf{P}}[-(\theta_i - \mathsf{P})^2 | v_i = A] - \mathsf{E}_{\mathsf{P}}[-(\theta_i - \mathsf{P})^2 | v_i = \varnothing] - c_i \\ &= \Pr[piv_{AB}] \left(-(\theta_i - A)^2 + (\theta_i - B)^2\right) - c_i, \end{aligned}$$

in which piv_{AB} denotes the event that voter i's ballot is pivotal in favour of A (see RO for more detail). That is, had he or she abstained, platform B would have been

[3] The literature generally presents the utility of a voter under the following shape: $P \times B + D - C$, in which P represents the pivot probability, B the benefit of changing the outcome of the election, D the sense of duty (or psychological costs of abstention), and C is the cost of going to the polls (say, opportunity cost of time). In our modelling, $c = C - D$, and the value of c may differ among voters.

implemented, whereas his or her additional vote changes the outcome, and platform A is implemented instead of B. Similarly, the value of a vote for B is given by:

$$\mathsf{E}\left[U\left(\theta_i, \mathsf{P}, B\right) - U\left(\theta_i, \mathsf{P}, \varnothing\right)\right] = \Pr\left[piv_{BA}\right]\left(\left(\theta_i - A\right)^2 - \left(\theta_i - B\right)^2\right) - c_i,$$

in which piv_{BA} denotes the event that voter i's ballot is pivotal in favour of B.

To shorten subsequent notations, define $W\left(\theta_i, v_i\right)$ as the value of action $v_i = A, B$ gross of the cost of voting:

$$W\left(\theta_i, v_i\right) = \Pr\left[piv_{v_i p}\right]\left(-\left(\theta_i - v_i\right)^2 + \left(\theta_i - p\right)^2\right), \text{ for } v_i, p \in \{A, B\}, \; v_i \neq p \qquad (2)$$

Using (2), the voter abstains if:

$$\max_{v_i \in \{A, B\}} W\left(\theta_i, v_i\right) \leq c_i, \qquad (3)$$

which gives us a simple rule to identify which voters turn out or abstain.

Now, let us turn our attention to the aggregate distribution of preferences in the population. I distinguish between left-wing voters with preferences $\theta_i \leq 0$ and right-wing voters with preferences $\theta_i \geq 0$. Setting $A < 0 < B$ and $A = -B$, all voters to the left of 0 prefer platform A to platform B, and conversely for right-wing voters. Also, define $\gamma_A \in (0, 1)$ as the fraction of the population that is left-wing and $\gamma_B = 1 - \gamma_A$ as the fraction of right-wing voters. Then, I assume that preferences are uniformly distributed over $[-\alpha, 0]$ and over $[0, \alpha]$. Put differently:

$$\begin{cases} \forall \theta_i \in [-\alpha, 0), & f(\theta_i) = \gamma_A / \alpha, \\ \forall \theta_i \in [0, \alpha], & f(\theta_i) = \gamma_B / \alpha, \\ \forall \theta_i \notin [-\alpha, \alpha], & f(\theta_i) = 0. \end{cases} \qquad (4)$$

This implies that $\int_{-\alpha}^{\alpha} f\left(\theta_i\right) d\theta_i = 1$ for any $\gamma_A \in (0, 1)$. Moreover, combining (1), (2) and the assumed positions of the parties yields:

$$\begin{cases} W\left(\theta_i, A\right) = 4 \Pr\left[piv_{AB}\right] \theta_i \, A \\ W\left(\theta_i, B\right) = 4 \Pr\left[piv_{BA}\right] \theta_i \, B. \end{cases} \qquad (5)$$

The size of the population, $\tilde{n}$, is also random and follows a Poisson distribution of argument λ: $\tilde{n} \sim \mathcal{P}(\lambda)$, where λ is the *expected* size of the population (see Myerson, 1997, 2000). Once the number of voters has been drawn, each voter is assigned a type θ_i by *i.i.d.* draws out of the piece-wise uniform distribution (4).

3 Equilibrium turnout

Now that the model has been presented, we can focus our attention on the problem at hand: how is turnout determined? To derive the equilibrium turnout level, we must solve for the (Bayesian Nash) Equilibrium of the voting game. That is, we have to find the equilibrium fraction of voters who prefer not to abstain, given the preferences of the electorate and the size of the population.

Clearly, a way to generate high turnout levels would be to assume that many voters "enjoy" voting. That is, one could find some reason why c_i (the cost of turning out for voter i) is non-positive for some fraction of the electorate. However, this would lead to no explanation for turnout, nor provide any rationale for turnout variations. Rather, this would mean that one needs to vary the modelled cost of voting *ex post* so as to make the fraction of voters with a negative cost of voting fit observed turnout, as Epstein and Mershon (see the opening quote of this paper) highlighted already. For this reason, I take a different approach and focus on positive voting costs. The goal of this approach is to derive turnout as a function of the characteristics of the election, not as a function of voters' preferences:[4]

Assumption 1 *All voters have positive (net) voting costs: $c_i = C_i - D_i \geq 0$, $\forall i$. Costs are idiosyncratic to each individual, and uniformly distributed between 0 and C:*

$$\tilde{c}_i \sim \mathcal{U}\left[0, C\right], \tag{6}$$

where $\mathcal{U}$ represents the uniform distribution and $C > 0$ is a parameter that represents the highest cost in the population, that is, $c_i \in \mathcal{C} = [0, C]$. Note that the assumptions of a uniform distribution and of non-negative voting costs are not necessary to derive the results. However, they greatly simplify subsequent computations.

Under assumption 1, the best response of a voter is characterized by a cost cut-off function such that voters with a cost lower than a given threshold turn out, while those with a cost higher than this threshold abstain:

Lemma 1 *Under assumption 1, for any* ***given*** *expected turnout, the probability that a voter turns out is proportional to his or her degree of extremism, $|\theta_i|$.*

Proof. By (3) and (5), a voter turns out if:

$$\begin{aligned} c_i &< W\left(\theta_i, B\right) = 4 \Pr\left(piv_{BA}\right) B\, \theta_i, \; \forall \theta_i \geq 0 \\ c_i &< W\left(\theta_i, A\right) = 4 \Pr\left(piv_{AB}\right) \left|A\, \theta_i\right|, \; \forall \theta_i \leq 0, \end{aligned}$$

where pivot probabilities depend on expected turnout (see below). For any given θ_i, under assumption 1, the probability that this happens is given by:

$$\begin{aligned} \Pr\left(c_i \leq W\left(\theta_i, B\right) | \theta_i \geq 0\right) &= \frac{4 \Pr\left(piv_{BA}\right) B}{C} \theta_i \\ \Pr\left(c_i \leq W\left(\theta_i, A\right) | \theta_i \leq 0\right) &= \frac{4 \Pr\left(piv_{AB}\right) |A|}{C} \left|\theta_i\right|. \end{aligned}$$

■

Clearly, this lemma only characterizes the best response of a voter for a given expected level of turnout, whereas the *equilibrium* level of turnout remains to be derived. To do so, one also needs to define when a vote can be pivotal:

Assumption 2 *The election rule is "first-past-the-post," and the implemented platform is A if $n_A \geq n_B$ and is B if $n_A < n_B$.*

[4] The same reasoning holds for other groups activities, like sport events. With soccer, for instance, a supporter will choose whether or not to attend a given match. Applied to this context, c_i represents the difference between the ticket price and the supporter's utility of 'being there,' whereas $W(\theta_i, v_i)$ relates to the impact of his supporting effort on the team's chances of winning. Setting $c_i < 0$, $\forall i$ then means that prices are low enough, and that uncertainty about the match outcome cannot influence attendance. Setting $c_i > 0$ for some supporters instead implies that more uncertain and/or crucial matches will benefit from larger attendance.

Note that assumption 2 introduces a tie-breaking rule that slightly differs from the standard "coin toss" assumption. This assumption is however made without loss of generality and is only meant to further simplify subsequent computations. Denote by s_A (respectively s_B) the fraction of the electorate that chooses action A (respectively B).[5] Using this definition, a direct result of assumption 2 is that:

Lemma 2 *Under assumption 2, the probability that a given ballot is pivotal in favour of B (respectively A) is given by:*

$$\Pr\left(piv_{BA}|s_A, s_B\right) = \frac{e^{-(\sqrt{s_A}-\sqrt{s_B})^2\cdot\lambda}}{2\sqrt{\pi\,\lambda}\,\sqrt[4]{s_A.s_B}} \tag{7}$$

$$\Pr\left(piv_{AB}|s_A, s_B\right) = \sqrt{\frac{s_B}{s_A}}\Pr\left(piv_{BA}|s_A, s_B\right). \tag{8}$$

Proof. Immediate from (20) in appendix 1. ■

From lemma 2, one can see that turnout probabilities are exponentially decreasing in turnout rates (which is defined as $t = s_A + s_B$) and in population size, λ. Moreover, the larger the difference between the vote shares of A and B (taken in absolute value), the smaller is this pivot probability. With the help of these two lemmas, one can show that:

Proposition 1 *Under assumptions 1 and 2, turnout can only be large if the two parties have (almost) equal support in the population. Moreover, the vote share of the winner is always smaller than its actual support in the electorate.*

Proof. From the definition of a Bayesian Nash equilibrium, the vote share of A, s_A, must be equal to the fraction of voters with preference $\theta_i \leq 0$ that turn out. That is, by lemmas 1 and 2:

$$\begin{aligned} s_A &= \int_{-\alpha}^{0} \frac{\gamma_A}{\alpha}\Pr\left(c_i \leq W\left(\theta_i, A\right)\right) d\theta_i \\ &= \frac{\alpha\,\gamma_A}{C}\sqrt{\frac{s_B}{s_A}}\frac{e^{-(\sqrt{s_A}-\sqrt{s_B})^2.\lambda}}{\sqrt{\pi\,\lambda}\,\sqrt[4]{s_A.s_B}}\,|A|\,. \end{aligned} \tag{9}$$

Similarly, the fraction of voters voting for B is:

$$s_B = \frac{\alpha\,\gamma_B}{C}\frac{e^{-(\sqrt{s_A}-\sqrt{s_B})^2.\lambda}}{\sqrt{\pi\,\lambda}\,\sqrt[4]{s_A.s_B}}\,B. \tag{10}$$

Computing the ratio of (9) over (10), I find:

$$\frac{s_A}{s_B} = \left(\frac{\gamma_A}{\gamma_B}\right)^{2/3} \quad (< \frac{\gamma_A}{\gamma_B},\ \forall\gamma_A > \gamma_B), \tag{11}$$

and hence that the vote share of the winner (compared to that of the party who loses the election) is smaller than its true support in the electorate.

[5] Note that $s_A + s_B$ need not sum to one, since a fraction $s_\varnothing \geq 0$ abstains.

Now, to derive total turnout, define the turnout rate $t = s_A + s_B$. Using (11) yields:

$$t = s_B \left(1 + \left(\frac{\gamma_A}{\gamma_B}\right)^{2/3}\right),$$

which can be used in (10) to obtain an implicit characterization of equilibrium turnout:

$$\left(1+\left(\tfrac{\gamma_A}{\gamma_B}\right)^{2/3}\right)^{-1} \times t = \frac{\alpha\,\gamma_B\,B}{C}\,\frac{\exp\left[-(\sqrt[3]{\tfrac{\gamma_A}{\gamma_B}}-1)^2\left(1+\left(\tfrac{\gamma_A}{\gamma_B}\right)^{2/3}\right)^{-1}\times t\lambda\right]}{\sqrt{\left(\tfrac{\gamma_A}{\gamma_B}\right)^{1/3}/\left(1+\left(\tfrac{\gamma_A}{\gamma_B}\right)^{2/3}\right)\times\pi\,\lambda\,t}}. \quad (12)$$

Quite clearly, since the exponential on the right-hand side of (12) is decreasing in $\left(\sqrt[3]{\frac{\gamma_A}{\gamma_B}}-1\right)^2 t$, the more γ_A differs from γ_B, the smaller turnout rate must be. Similarly, the larger is λ the smaller is t. ■

Proposition 1 illustrates the apparent failure of the rational-instrumental theory of voting to explain turnout. Its first result indeed shows that, except for the case of tied parties, the act of turning out should be motivated by other rationales than the voters' desire to influence the outcome of the election. Where does this failure come from? Simply, since the act of turning out is only valuable when one's vote is pivotal, and since the probability of being pivotal decreases in the number of voters who participate in the election, the incentives to participate fade out very quickly when the electorate is large. Moreover, the probability that parties are tied is even smaller if one of the parties has much stronger support than the other party. Figure 1 illustrates this result for $B = 10 = -A$, $\alpha = C = 1$, and three values of λ. As one can see from this figure, the larger is λ, the smaller is the turnout rate and the smaller is the range of values of γ_A for which turnout can be large.

<< Insert Figure 1 about here >>

Quite clearly, such a result is highly unsatisfactory, and explains quite well why the rational-instrumental approach to voting has been perceived extremely weak. The goal of the next sections will thus be to show that this apparent weakness actually results from wrongly specified underlying assumptions, and hence that the rational-instrumental approach has more to tell us than proposition 1 suggests.

4 Uncertain preferences

As stated in the introduction, a crucial aspect that one must consider to improve the predictive power of the rational-instrumental approach is to take the effect of the mandate into account. Yet, there is another implicit assumption that I made until now, and which may have some weight on the results of proposition 1. Namely, that voters exactly know the distribution of preferences in the electorate. That is, voters compute their probability of being pivotal given γ_A and γ_B. An alternative (and more realistic) assumption would be:

Assumption 3 *The share of voters having left-wing and right-wing preferences is random and is unknown to the voters prior to the election. I.e., voters only have priors on the* distribution *of* γ_A *and* γ_B*:*

$$\begin{aligned} \tilde{\gamma}_A &\sim \mathcal{L}(\bar{\gamma}_A) \\ \tilde{\gamma}_B &= 1 - \tilde{\gamma}_A \end{aligned}$$

That is, the share of voters supporting A is random, with an average $\bar{\gamma}_A$. Note that the distribution of $\tilde{\gamma}_A$ needs not be specified, and I leave it to be distributed under some "law" $\mathcal{L}$.

Why assuming random distributions? Why is the assumption of fixed-and-known margins necessarily wrong? Strategic positioning by parties somewhat justifies that expected shares should be around 50%. But, if the distribution of preferences were fixed, opinion polls should leave no doubt about the identity of the winner of the election. Instead, uncertainty remains most of the time. For instance, two weeks before the 1997 election in the U.K., bookmakers were taking bets at 4 to 1 for John Major in his fight against Tony Blair, while the latter was expected to have a lead of 10%. With a Poisson distribution and fixed expected voting shares, one can easily compute the number of votes such that the leader is elected with a probability of 99.9%:

Table 1.

Total number of votes $\bar{n}$ that ensures victory with a probability of 99.9% when the expected share is:				
50.1%	50.5%	51%	52%	55%
$\bar{n} = 2,427,000$	$\bar{n} = 97,000$	$\bar{n} = 24,000$	$\bar{n} = 6,000$	$\bar{n} = 975$

That is, with 975 votes, the odds against Major should have been at 1000 to 1, and much higher with a larger turnout. One readily sees that, for a "standard" turnout, i.e. a turnout of a few million votes, the assumption of known margins is incompatible with the slightest uncertainty on the identity of the winner, unless expected shares are exactly 50%.

Yet, one can easily derive turnout under the alternate specification of assumption 3. It substantially differs from the results of proposition 1:

Proposition 2 *Under assumptions 1-3, and if* $\Pr(\tilde{\gamma}_A \simeq \tilde{\gamma}_B) > 0$, *absolute turnout is proportional to* $\lambda^{2/3}$ *and the turnout* **rate** *decreases in the cubic root of population size:* $\bar{n}/\lambda \propto 1/\sqrt[3]{\lambda}$.

Proof. From Myerson (2000)'s *Magnitude Theorem*, we know that the distribution of states of the world *conditional on being pivotal* degenerates to a point mass around the set of states of the world that make pivotability most likely. That is, if the voter wants to assess his or her probability of being pivotal in the election, he or she must only consider the states of the world where pivotability is most likely. Hence, by (7) and (8), only values of $\tilde{\gamma}_A$ close to 50% must be considered. The main determinant to turnout is thus the probability of the event $\tilde{\gamma}_A \simeq 1/2$:

$$\begin{aligned} W(\theta_i, A) &= 4 \ \Pr(piv_{AB}) \ |A\,\theta_i| \\ &\simeq 4 \ \Pr[\tilde{\gamma}_A \simeq 1/2] \cdot \frac{1}{\sqrt{2\pi\,\bar{n}}} \cdot |A\,\theta_i| \,, \ \forall \theta_i \leq 0; \end{aligned}$$

$$\begin{aligned} W(\theta_i, B) &= 4 \ \Pr(piv_{BA}) \ B\,\theta_i \\ &\simeq 4 \ \Pr[\tilde{\gamma}_A \simeq 1/2] \cdot \frac{1}{\sqrt{2\pi\,\bar{n}}} \cdot B\,\theta_i, \ \forall \theta_i \geq 0, \end{aligned}$$

where $\bar{n}$ is the equilibrium number of voters who turn out, which remains to be determined. By lemma 1, voters who turn out are the ones who have a sufficiently low cost of voting. Assuming the expected size of the population (λ) to be sufficiently large, the equilibrium is defined as the value of $\bar{n}$ that solves:

$$2\lambda \int_0^{\alpha} \frac{4 \ \Pr[\tilde{\gamma}_A \simeq 1/2] \cdot B\,\theta_i}{C\sqrt{2\pi\bar{n}}} d\theta_i = \frac{4\lambda \Pr[\tilde{\gamma}_A \simeq 1/2]}{\sqrt{2\pi\bar{n}}\,C} B\,\alpha^2 = \bar{n}.$$

Solving for $\bar{n}$ then yields:

$$\bar{n} = \left(\frac{8\lambda^2}{\pi\,C^2} \Pr[\tilde{\gamma}_A \simeq \tilde{\gamma}_B]^2 \ B^2\,\alpha^4 \right)^{1/3},$$

and the participation *rate* is thus given by:

$$\frac{\bar{n}}{\lambda} = 2 \left(\frac{\Pr[\tilde{\gamma}_A \simeq \tilde{\gamma}_B]^2 \ B^2\,\alpha^4}{\pi\,C^2\,\lambda} \right)^{1/3}.$$

Consequently, for continuous distributions of $\tilde{\gamma}_A$, forecasted turnout will be rather low (compared to observations), but increasing in the probability of close margins, that is in $\Pr(|\tilde{\gamma}_A - \tilde{\gamma}_B| < \varepsilon)$, with $\varepsilon \underset{\bar{n}\to\infty}{\longrightarrow} 0$. ■

Hence, introducing uncertainty on the preferences of the electorate shows that total turnout can increase rather quickly in population size. This is, in a sense, the positive lesson of the exercise. However, there is also a negative lesson; namely, that turnout is only large when voters believe that, with a high enough probability, parties have (almost) equal support in the electorate: $\Pr[\tilde{\gamma}_A \simeq 1/2]$ must be sufficiently large. It thus appears that, even though introducing assumption 3 does improve the results, one still needs an extra step to reconcile the model with empirical regularities. Hence, I now turn to a discussion of assumption 2, which will prove much less innocuous than it seems at first glance.

5 If mandate matters

What are the effects of the mandate on a voter's incentives to turn out? Under assumption 2, the utility of a voter i is given by:

$$\begin{aligned} U(\mathsf{P}, \theta_i | n_A < n_B) &= U(B, \theta_i) \\ U(\mathsf{P}, \theta_i | n_A \geq n_B) &= U(A, \theta_i). \end{aligned}$$

As one can read from these expressions, the utility of any given voter can thus only take two values, with a discrete jump in $n_A = n_B$. However, if ex-post policy-making involves a bargaining of some sort once the election is over, or if the elected candidate/party pays attention to the share of votes he obtained, the implemented platform may not be an "all-or-nothing" composition of the proposed platforms. For instance, Ronny Razin (2000) demonstrates how this vote share actually communicates information to the candidates, who consequently have an incentive to moderate their policy when their margin of victory shrinks.[6]

When this happens, a lot of "activity" happens around the change in majority, i.e. policy changes can be substantial even when $n_A \neq n_B$. In other words, the implemented platform will be a smoother function of the vote shares received by the parties than under assumption 2. Accordingly, I take the implemented policy to be a convex combination of the share of votes received by the two parties: define n_B (resp. n_A) as the realized number of votes for party B (resp. A) and $\bar{n} = n_A + n_B$ as the realized turnout. I pose:

Assumption 2' *The implemented platform,* P, *is a continuous function of the share of votes going to the two parties, such that* $A \leq \mathsf{P}\left(\frac{n_B}{\bar{n}}\right) \leq B$, *and* $\mathsf{P}'(.) > 0$.

More precisely,

$$\mathsf{P}\left(\frac{n_B}{\bar{n}}\right) = \left[1 - \sigma\left(\frac{n_B}{\bar{n}}\right)\right] \cdot A + \sigma\left(\frac{n_B}{\bar{n}}\right) \cdot B \tag{13}$$

where σ is a "power sharing function," in the spirit of Stigler (1972) or Grillo and Polo (1993), with $\sigma \in [0,1]$, continuous, differentiable, and where σ' is positive and reaches its maximum in 1/2. An example is illustrated in Figure 2.

<<Insert Figure 2 about here >>

Denote the realized vote shares of the two parties by $s_p \equiv n_p/\bar{n}$, $p = A, B$. For $\bar{n}$ expressed votes, a voter will change the share of votes going to party B by:

$$\begin{aligned} \frac{n_B}{\bar{n}+1} - \frac{n_B}{\bar{n}} &= -\frac{s_B}{\bar{n}+1}, \text{ if her vote goes to } A \\ \frac{n_B+1}{\bar{n}+1} - \frac{n_B}{\bar{n}} &= \frac{s_A}{\bar{n}+1}, \text{ if her vote goes to } B. \end{aligned}$$

Therefore, for any given $\bar{n}$ and n_B, the *expected effect of a vote* on the implemented platform is:

$$-\mathsf{P}'(s_B) \times \tfrac{s_B}{\bar{n}+1}, \quad \text{if the vote goes to } A \tag{14}$$

$$\mathsf{P}'(s_B) \times \tfrac{s_A}{\bar{n}+1}, \quad \text{if the vote goes to } B. \tag{15}$$

Now, I need to derive the expected effect of a vote in case n_B and $\bar{n}$ are random. Quite clearly, $s_B = n_B/\bar{n}$ is a priori not independent of $\bar{n}+1$, which makes it difficult to derive the values of (14) and (15). However, the following lemma demonstrates that the different elements in (14) and (15) are asymptotically independent one from another:

Lemma 3 *If turnout,* $\bar{n}$, *is distributed according to a Poisson distribution of argument* λ *and* $\tilde{n}_A \sim \mathcal{P}(\gamma_A.\lambda)$, $\tilde{n}_B \sim \mathcal{P}(\gamma_B.\lambda)$, *with* $0 < \gamma_A = 1 - \gamma_B < 1$, *then*
1) If σ *is linear, there is* regressive *independence between the number of votes cast and the expected effect of a vote;*
2) For any twice continuously differentiable function σ, *there is asymptotic* stochastic *independence between the expected number of votes* $\bar{n}$, *and the marginal effect of a vote* $\mathsf{P}'(s_B) \times s_p$.

[6] Other types of equilibria can also arise. See Razin (2000) for more detail.

Proof. For any random X and Y, we know that

$$\mathsf{E}[XY] = \int_{-\infty}^{+\infty} Y \; f(Y)\, \mathsf{E}(X|Y)\, dY,$$

where $f(Y)$ is the density of Y. Applying this expectation operator to (14), let $X = a \cdot n_B/\bar{n}$ and $Y = 1/(\bar{n}+k)$. Holding Y constant (that is, holding $\bar{n}$ constant), the distribution of n_B is a binomial $B(\gamma_B, \bar{n})$. In that case, $\mathsf{E}[n_B/\bar{n}|Y] = \gamma_B$ independently of $\bar{n}$. Therefore, $\mathsf{E}[X|Y]$ is independent of Y if X is a linear function of $n_B/\bar{n}$. In statistics, this is saying that the share of B is *regressively* independent of the number of votes. However, it is also clear that $n_B/\bar{n}$ is not *stochastically* independent of the number of votes. For instance, if $\bar{n} = 2$, the share of B can be only 0, 0.5 or 1. Instead, the realization $n_B/\bar{n} = 0.5$ is impossible to obtain when $\bar{n}$ is odd. However, the *De Moivre-Laplace limit theorem* (see Mood et al., 1974, pp120-121), tells us that, for $\bar{n}$ large and $\forall \varepsilon > 0$, we have:

$$\Pr\left(\gamma_B - \varepsilon \leq n_B/\bar{n} \leq \gamma_B + \varepsilon\right) \simeq 2\,\Phi\left(\frac{\varepsilon\sqrt{\bar{n}}}{\sqrt{\gamma_B\,(1-\gamma_B)}}\right) - 1 \underset{\bar{n}\to\infty}{\longrightarrow} 1,$$

where Φ is the C.D.F. of the centered-reduced normal distribution. In that case, for all twice-continuously differentiable functions σ:

$$\int_{-\infty}^{+\infty} \phi(n_B/\bar{n}|\bar{n})\, \sigma\left(n_B/\bar{n}\right)\; d\left(n_B/\bar{n}\right) \underset{\bar{n}\to\infty}{\longrightarrow} \sigma\left(\gamma_B\right),$$

where $\phi\left(.\right)$ is the P.D.F. of the centered-reduced normal distribution. ■

This lemma has a very simple meaning: if the power sharing function is continuous, in contrast to the results of RO, a vote can always affect the implemented platform at the margin, and the voter can consider two effects separately. On the one hand, the more voters turn out, the smaller the effect of his or her own ballot on the implemented platform. On the other hand, the voter can evaluate whether to vote for A or for B only by looking at the distribution of preferences in the electorate (γ_B). The former and the latter are stochastically independent one from another.

Of course, if the function σ had a discontinuity in $n_A = n_B$, one would also have to consider the probability of a tie, as in previous sections. In this case, turnout would increase both in the expected effect of a vote *and* in the probability that the two parties are tied. Another point highlighted by this lemma is that, at the margin, the effect of a vote only depends on the shape of $\mathsf{P}\left(s_B\right)$. So to say, the effect of a vote becomes deterministic and not probabilistic, which means that the value of a vote is much higher than under previous specifications:

Proposition 3 *Under assumptions 1, 2' and 3, total turnout is proportional to the square root of expected population size (t is proportional to $1/\sqrt{\lambda}$) and increases in the expected value of* $|\mathsf{P}'\left(.\right)|$.

Proof. See appendix 2. ■

What do we learn from this proposition? First, that instrumental voting is compatible with large turnouts when mandate matters, even if the probability of being pivotal is minimal in the commonly accepted sense. Second, if the power sharing function is similar to that in Figure 2, turnout falls smoothly in the share of the winner (and not abruptly as in Section 2). Third, this proposition shows that the importance of the election is directly linked to the shape of the power sharing function, for which accurate empirical estimates are not available. Yet, it suggests that the turnout decision in first-past-the-post elections should follow the same decision rules as in proportional representation elections.

It is also interesting to note that, under the assumption of concave utility functions, there is an *underdog* effect on the day of the elections. That is, if a given share of the population, say $\gamma_A > 0.5$, prefers party A to party B, election results will tend to give A a fraction of the votes that is lower than γ_A. To see this, assume for a moment that γ_A is fixed and certain. By (14) and (15), the expected effect of a vote for A is smaller than the expected effect of a vote for B. Therefore, *ceteris paribus,* the value of a vote for A is smaller than that of a vote for B. Moreover, this cumulates with a lower interest in the election for the majority, as the average distance between the A-voters and the platform is smaller. Hence, voters on the side of the loser will have higher turnout rates than voters supporting the expected winner of the election, and vote results will be closer to parity than the true preferences of the population.

6 Numerical simulations

The goal of the previous sections was to show that assumptions oft-made in the literature were largely responsible for the model's lack of predictive power, while relaxing them makes the model more realistic and allows it to predict a voting behavior that is closer to actual observations. Yet, the above developments led to rather abstract results. The aim of this section is to make them more concrete, by applying the results of proposition 3 to actual data. I run two sets of simulations. First, I apply my theoretical results to U.S. presidential elections since 1960. Second, I check whether the results are consistent with cross-country variations in turnout rates.

6.1 The secular fall in U.S. turnout rates

U.S. turnout rates are the most studied in the literature. As evidence shows, they display a clear negative trend: turnout rates fell regularly since the end of World War II (empirical regularities are summarized in appendix 3), and yet this fall remains largely unexplained. Our lack of understanding thus makes it tempting to attribute this fall to exogenous changes in the tastes of the electorate: either new generations feel less involved in politics or their opportunity cost of time has increased. Under both explanations, the presumed culprit is a regular and steady increase in the net costs of voting over time.

Yet, if voters rationally decide how much information to acquire, and if the size of the electorate increases, growing disinterest in politics might only reflect a rational decision to abstain with a higher probability. If this alternate interpretation holds, however, the fall in turnout rates cannot be attributed to a change in the intrinsic tastes of the electorate, but instead to a fall in each voter's incentives to participate in elections.

To check whether this rationalist interpretation holds, I apply the results of proposition 3 to the data made available by the Federal Election Commission on their website (www.fec.gov) for U.S. Presidential elections between 1960 and 2000. Simulations are performed as follows. First, I compute the actual turnout rate for every presidential election since 1960, using the number of registered voters ($\lambda_t = \#Registered_t$) to represent total population size, since unregistered people cannot vote:[7]

$$T_t^{Actual} = \frac{\#Ballots\,Cast_t}{\lambda_t}.$$

Then, I derive the turnout level predicted by proposition 3:

$$T(\lambda_t) = \sqrt{2\frac{K}{C\,\lambda_t}}, \tag{16}$$

which clearly depend on the ratio K/C, where K is the voters' average valuation the *expected effect of their vote* (see appendix 2), and C is the highest cost of voting in the population ($c_i \sim \mathcal{U}[0, C]$). Since predicted turnout depends on this ratio, for which no estimate is available, I first calibrate the model on the actual turnout rate in 1960. That is, the value of K/C is chosen to make the predicted and actual values of T_{1960} coincide ($T_{1960}^{Actual} = 93\%$). This value of K/C is then maintained constant over time, and I only let population size vary. The results of this exercise are displayed in Figure 3.

<< Insert Figure 3 about here >>

As Figure 3 illustrates, a rationalist interpretation is largely consistent with the actual evolution of turnout after 1960. For instance, as the graph shows, the fall in turnout rates accelerated in 1972, and a similar acceleration is predicted by the model. Why? Simply because the children of the 1950s' Baby Boom reached the age of voting around that year, and the size of the voting age population consequently increased by 17% between 1968 and 1972, instead of 4 or 5% in the preceding elections. By contrast, the largest prediction error occurs in 1992. In that year, however, 19% of the electorate voted for Ross Perot, which could be interpreted as a "virtual reduction" in the size of the electorate that participates in the election between Bill Clinton and George Bush. To take this "Perot effect" into account, I compute the turnout predicted by the model in case only 81% of the population existed at the time of the 1992 election. The result of this exercise is represented by the dashed line in Figure 3, and is shown to explain the deviation quite well.

According to this simulation, the "secular fall in turnout" can thus be attributed to strategic reasons alone. The larger the size of the electorate, the smaller is the value of a ballot, and thus the lower turnout should be. In the case of the U.S., the number of registered voters having more than doubled over the period, turnout rates are predicted to fall by about $1/\sqrt{2}$.

Note that, in line with the modelling strategy of the paper, this simulation assumes that all voters are purely strategic, and that all voters have a positive cost of voting. Yet, empirical evidence suggests that some fraction of the electorate have a negative (net) cost of voting –meaning that their sense of democratic duty is larger than their cost of going to the polls. According to Brody and Page (1973), for instance, about 43% of the population in 1968 had negative (net) costs of voting. Proposition 3 can easily be extended to negative voting costs,[8] but simulations generate weaker results

[7]Two remarks are in order. First, actual registration data are missing for 13 states in 1960 and 9 states in 1964. The registration rates used for these two years are thus partially extrapolated. Second, *registration costs* may have varied between 1960 and 2000. However, these cannot influence simulation results, since I focus on the number of already registered voters.

[8]Theoretical developments and simulation results are available from the author upon request.

under this specification: actual turnout dropped by 26 points between 1960 and 2000, whereas the model only predicts a drop of 20 points. However, participation also fell in other kinds of social activities, as Putnam (1995) documents in detail. According to his study, American citizens became increasingly less inclined to participate in group activities, which, he argues, might result from the dissemination of television sets across the country. Unsurprisingly, if I also let the share of voters with negative costs fall over time, the model then recovers all its explanatory power.

Two competing explanations can thus be put forward to explain why turnout fell so much over this 40-year period. The former is that voters are purely strategic. The alternate one combines strategic motivations with the fact that voting costs have increased over time. Clearly, the second explanation is empirically unfalsifiable, since voting costs can only be measured ex post, through voters' behaviour. Yet, there are two reasons why I deem the first explanation as more satisfactory. First, television may indeed have increased the opportunity cost (or decreased the relative benefit) of participating in social activities. However, television broadcasts also increase voters' awareness of the electoral campaign, and this rising amount of information should instead increase turnout (for evidence on the positive influence of radio on turnout, see Strömberg, 2001). Second, if voting costs had substantially increased, voters' registration rates should also have fallen (a specificity of the U.S. being that registration is voluntary). Instead, it increased over the same period.

6.2 Cross-country comparisons

Performing a similar exercise for other democracies proves much more difficult. The U.S. political environment is indeed unusually stable by international standards: only two parties consistently dominate the political scene, and the case of R. Perot showed that turnout rates are quite sensitive to the introduction of additional challengers. Accordingly, turnout rates are much more volatile in other democracies. Secondly, the size of the electorate has increased more significantly in the U.S. than in many other mature democracies.[9] Therefore, tracking the effect of population size on turnout for every election will have a less significant impact in other countries.

For these reasons, this cross-country analysis will not try to track the evolution of turnout rates on an election-by-election basis. Instead, I apply proposition 3 to the *long-run* evolution of turnout rates, that is on the cumulated fall in turnout rates between 1960 and the most recent legislative elections. The sample is limited by the small number of long-established democracies: I focus on the fifteen European Union countries, plus six other major democracies.[10] This exercise should thus not be seen as an exhaustive empirical test of the model, but instead as a first attempt to shed some light on the model capability of tracking these evolutions. A more complete test would require a much more detailed institutional analysis, which is beyond the scope of this paper.

[9] For instance, between 1960 and the most recent elections, voter registration increased more dramatically in the U.S. (+113%) than in France (+49%), Italy (+44%), the UK (+25%), Belgium (+21%) or Austria (+20%). (Data from IDEA, `www.idea.int`)

[10] The other countries included in the sample are: Australia, Canada, Iceland, Japan, Norway, and the US. For Spain, Greece and Portugal, I consider the first year in which democratic attainments are sufficient (a Freedom House index smaller or equal to 3, i.e. 1981 for Greece, 1982 for Spain and, 1983 for Portugal).

Another problem is that cultural differences, as well as differences in electoral systems, are also likely to affect voting behaviour: voting costs and local tastes for "social activities" may differ across countries, and the model cannot explain such differences. I thus apply the model under two different scenarios. *Scenario 1* narrowly applies theoretical results on the data: in line with the above simulations, this scenario assumes a common distribution of (non-negative) voting costs in all countries: c_i follows the same uniform distribution $\mathcal{U}[0, C]$ in all countries. *Scenario 2*, instead, allows the distribution of voting costs to differ across countries: the distribution of costs in country j is given by $c_i^j \sim \mathcal{U}\left[-\gamma_j / \left(1-\gamma_j\right) C\,,\, C\right]$, such that, in country j, a fraction γ_j of voters has negative voting costs.[11]

Under the two scenarios, the ratio K/C is left to be identical for all countries, and I derive the fall in turnout predicted by the model under each scenario. Then, to test the quality of this prediction, I regress by OLS the actual evolutions of turnout on the predicted ones. The quality of the prediction is highest if, in this regression, the intercept is zero, and the coefficient associated with predicted turnout is equal to one.

The results are displayed in Table 2. In all regressions, the endogenous variable is $T_{recent}^{Actual}/T_{1960}^{Actual}$, that is: the ratio of the recent turnout rate over the one around 1960. To give some additional information on the data sample, the first column in Table 2 displays the results of the "raw" regression where turnout changes are only explained by the increase in the number of registered voters, $\lambda_{Recent}/\lambda_{1960}$. The second column provides the results obtained under the first scenario, and the third column provides the ones obtained under the second scenario.

Table 2: cross-country turnout variations.

Endogenous variable: $\left(T_{recent}^{Actual}/T_{1960}^{Actual}\right)_j$			
Intercept	1.1**	0.37*	−0.04
	(0.10)	(0.15)	(0.12)
$\lambda_{Recent}/\lambda_{1960}$	−0.13		
	(0.67)		
Predicted - Scenario 1		0.59**	
		(0.16)	
Predicted - Scenario 2			1.03**
			(0.13)
R^2	*0.16*	*0.41*	*0.76*
Note: Standard errors in parentheses. *: significant at 5%, **: significant at 1%			

Table 2 thus shows that the predictions of the model are consistent with the data (Column 2), even though the raw correlation between turnout changes and population growth is not significant (Column 1). Moreover, using an "informed prior" about the relative tastes of the populations in different countries (e.g. that participation in social activities tends to be higher in Nordic European countries) noticeably improves the predictive power of the model (Column 3).

[11] I set $\gamma_j = 0$ in the U.S., the U.K., and Japan. In continental European countries and in Canada, I set $\gamma_j = 0.3$, since social activities seem culturally more important in these countries. Similarly, I set $\gamma_j = 0.6$ in Nordic European countries, and $\gamma_j = 0.9$ in countries where voting is compulsory and abstention fined (Australia and Belgium). The quality of the results (see Table 2 below) is marginally reduced if I set the γ_j's to *e.g.:* 0, 0.4, 0.6 and 0.8 respectively, or if $\gamma_{\text{Canada}} = 0$.

In my view, this evidence gives weight to the results of proposition 3. Yet, these simulation results only apply to *changes* in turnout rates. If instead I try to explain turnout *levels*, the predictive power of the model is much lower: the raw correlation between turnout levels and population size is around 40%, and the model does not increase this value substantially. In other words, and in contrast with the results for the U.S., this shows that other than purely strategic factors also influence turnout levels. Still, the results of Table 2 suggest that these factors do not vary noticeably across time: they are important to explain why turnout is higher or lower than predicted in each country, but have little influence on *changes* in turnout rates over time.

Of course, one of the factors that may explain the unexplained cross-country variance in turnout levels might be the differences in electoral systems. However, according to the model, first-past-the-post (FPTP) and proportional representation systems should not behave much differently, since the margin of victory also affects the implemented platform in the FPTP system. Here again, a more detailed institutional analysis would be helpful. Yet, in line with this presumption, regressions show that the average prediction error of the model is close to zero in both electoral systems, either when I regress turnout *levels* on electorate size or turnout *changes* on electorate growth.

7 Conclusions

This paper has assessed the real limits of the model of instrumental voting. It showed that, in contrast to usual perceptions, a rational decision by the voters may explain several stylized facts regarding turnout. Even though "standard" models fail to explain empirical regularities, the instrumental voting hypothesis is thus not at the roots of the problem. I have argued that, on the contrary, it remains one of the important forces behind the turnout decision. More precisely, I have shown that some unjustifiable assumptions are generally made, and that they alone are responsible for the failures of the model to reproduce those empirical regularities.

More precisely, the paper showed that:

1. The shares of votes going to each party cannot be modelled as fixed.
2. The effect of victory margins on implemented policy must be taken into account.

I have also shown that the theoretical predictions of this extended model are consistent with the downward trend in turnout rates in the U.S. since 1960, and with the cross-sectional evolution of turnout rates in major democracies throughout the world.

In other words, the instrumental voting theory altogether may have tended to be rejected, not because it is "*highly suspect in nature*," as Fiorina (1997, p403) puts it, but rather because of the excessive simplifications used in those models. It should be stressed, however, that my simulation results focus on the *dynamics* of turnout rates: the fall in turnout rates is consistent with the increasing size of the population. Yet, the other arguments underlined in the literature, such as information, peer pressure, sense of duty, or the willingness to participate in social activities, also remain important in explaining turnout *levels*. The instrumental and non-instrumental approaches thus appear to be complementary, but not substitutes, in determining the voters' turnout decision.

Beyond these observations, the theoretical results in the paper also have an empirical reach. Their implications may seem to contradict some empirical findings, and the reader may thus reject them on these grounds. Such a shortcut is however incorrect, since most of the existing empirical evidence cannot be used to test these results. For instance, I claimed that, if the implemented policy is a function of the "mandate" given to the winner, closeness does not matter in the same way as in the standard Riker and Ordeshook (1968) model. In order to test this prediction, one might be tempted to compare turnout rates during presidential and mid-term elections, since policy changes might be more "continuous" after mid-terms than after presidential elections. Hence, according to my results, mid-term elections should generate *higher* turnout rates than presidential elections, whereas empirical evidence shows that the opposite holds true. Still, it is also true that potential policy changes are more limited after mid-term elections, and this must decrease equilibrium turnout. Additional empirical research is thus needed to further test these results, because existing empirical evidence is somewhat orthogonal to the questions raised by the paper (for instance, an accurate assessment of platform changes after mid-terms and after presidential elections is needed to test the model). Another point raised by the empirical literature is whether closeness increases or decreases turnout. If we believe in the power-sharing function proposed in Section 5, the large expected victory of a candidate should lower turnout. However, another effect goes in the opposite direction: if large electoral support reflects larger expected gains from electing the winner, turnout can be increased. Again, without additional empirical work, we cannot tell in which direction turnout should go.

Summing up, the theoretical contribution of this paper points to micro-founded ways to pursue future empirical research: first, insofar as the trade-offs analyzed in this paper apply to other activities than elections (general assembly meetings in corporations, sport events,...), additional insight could be gained by not focusing on elections exclusively. Second, with regard to democratic elections, if we could carefully assess which parts of the population are more targeted by parties, what are the potential gains for different classes of the population, how uncertain the outcome of the election is, and so on, we would have better estimates of the expected value of a vote and, from there, use the results of this paper to derive potential turnout rates.

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Appendices

Appendix 1: General properties of Poisson games

1.1. The Poisson distribution

A variable $\tilde{n}$ is distributed according to a Poisson distribution of parameter λ if:

$$\Pr(\tilde{n} = k) = \frac{e^{-\lambda} \times \lambda^k}{k!}, \quad (k \in \mathcal{N}). \tag{17}$$

It follows that, for $\tilde{n}_A \sim \mathcal{P}(\lambda_A)$ and $\tilde{n}_B \sim \mathcal{P}(\lambda_B)$:

$$\begin{aligned} \Pr(\tilde{n}_A = \tilde{n}_B + c) &= \sum_{k=0}^{\infty} \Pr(\tilde{n}_A = k + c) \Pr(\tilde{n}_B = k) \\ &= \sum_{k=0}^{\infty} \frac{e^{-\lambda_A - \lambda_B} . \lambda_A^{k+c} . \lambda_B^k}{(k+c)!\ k!}. \end{aligned} \tag{18}$$

1.2. Using Bessel functions in pivot probabilities

The definition of a modified Bessel function I of degree c is given by:

$$I_c(2z) = \sum_{k=0}^{\infty} \frac{z^{2k+c}}{k!\ (k+c)!}, \text{ with } \lim_{z \longrightarrow \infty} I_c(z) = \frac{e^z}{\sqrt{2\pi z}}. \tag{19}$$

Rewriting (18) thus obtains:

$$\begin{aligned} \Pr(\tilde{n}_A = \tilde{n}_B + c) &= e^{-\lambda_A - \lambda_B} \frac{\left(\sqrt{\lambda_A}\right)^c}{\left(\sqrt{\lambda_B}\right)^c} \sum_{k=0}^{\infty} \frac{\left(\sqrt{\lambda_A}\right)^{2k+c} . \left(\sqrt{\lambda_B}\right)^{2k+c}}{(k+c)!\ k!} \\ &= e^{-\lambda_A - \lambda_B} \left(\lambda_A / \lambda_B\right)^{\frac{c}{2}} I_c\left(2\sqrt{\lambda_A . \lambda_B}\right) \\ &\underset{\lambda_A, \lambda_B \to \infty}{\rightarrow} e^{-\left(\sqrt{\lambda_A} - \sqrt{\lambda_B}\right)^2} \left(\lambda_A / \lambda_B\right)^{\frac{c}{2}} \left(2\sqrt{\pi}\sqrt[4]{\lambda_A . \lambda_B}\right)^{-1}. \end{aligned}$$

So, if $\lambda_i = s_i \cdot \lambda$, one obtains:

$$\Pr(\tilde{n}_A = \tilde{n}_B + c) \simeq e^{-\left(\sqrt{s_A} - \sqrt{s_B}\right)^2 \lambda} \left(s_A / s_B\right)^{\frac{c}{2}} \left(2\sqrt{\pi\lambda}\sqrt[4]{s_A . s_B}\right)^{-1}. \tag{20}$$

That is, when $s_A = s_B$,

$$\Pr(\tilde{n}_A = \tilde{n}_B + c) \simeq \left(2\sqrt{\pi\lambda}\sqrt[4]{s_A . s_B}\right)^{-1}, \tag{21}$$

which decreases at speed $\sqrt{\lambda}$. By contrast, when the s_i's differ, the probability decreases exponentially towards 0.

Appendix 2: Proof of proposition 3

Exploiting the results of lemma 3, the expected utility of a voter who abstains is given by:

$$\begin{aligned} \mathsf{E}U\left(\theta_i, \mathsf{P} | v_i = \varnothing\right) &= \mathsf{E}_{s_B}\left[-\left(\theta_i - \mathsf{P}\left(s_B\right)\right)^2\right] \\ &= -\theta_i^2 + \int_0^1 f_{s_B}\left(s_B\right) \cdot \mathsf{P}\left(s_B\right) \cdot \left(2\theta_i - \mathsf{P}\left(s_B\right)\right) \cdot ds_B, \end{aligned} \tag{22}$$

where s_B is the share of votes cast on B and $f_{s_B}(s_B)$ is the density of s_B. By (15) and lemma 3, for $\bar{n}$ sufficiently large, the value of an additional vote for B can be computed using the derivative of (22) with respect to n_B:

$$\begin{aligned} W(\theta_i, B) &\simeq \mathsf{E}_{\bar{n}} \frac{2}{1+\bar{n}} \int_0^1 f_{s_B}(s_B) \cdot \mathsf{P}'(s_B) \cdot (1-s_B) \cdot (\theta_i - \mathsf{P}(s_B)) \cdot ds_B \qquad (23) \\ &\simeq 2\,\Omega^B(\theta_i)\,/\,\bar{n}, \qquad (24) \end{aligned}$$

where $\Omega^B(\theta_i)$ is equal to the integral in (23).

Following the same reasoning, a vote for A is worth:

$$\begin{aligned} W(\theta_i, A) &\simeq \mathsf{E}_{\bar{n}} \frac{2}{1+\bar{n}} \int_0^1 f_{s_B}(s_B) \cdot [-\mathsf{P}'(s_B)]\; s_B\; (\theta_i - \mathsf{P}(s_B)) \cdot ds_B \\ &\simeq 2\,\Omega^A(\theta_i)\,/\,\bar{n}. \qquad (25) \end{aligned}$$

The Nash response of voter i will thus be:

$$\begin{aligned} &\text{Play } A \text{ if } W(\theta_i, A) > \max[W(\theta_i, B), c_i], \\ &\text{Play } B \text{ if } W(\theta_i, B) > \max[W(\theta_i, A), c_i], \\ &\text{Abstain if } \max[W(\theta_i, A), W(\theta_i, B)] \le c_i. \end{aligned}$$

Knowing the Nash responses of each voter, one can solve for the Bayesian Nash Equilibrium of the game. From (24) and (25), the value of turning out is inversely proportional to $\bar{n}$, the expected turnout. The Bayesian Nash Equilibrium of the game is thus obtained when $\bar{n}$ is consistent with the Nash responses of all voters:

$$\lambda \cdot \int_{-\alpha}^{\alpha} \Pr\left[c_i \le \frac{2}{\bar{n}} \max\left\{\Omega^A(\theta_i), \Omega^B(\theta_i)\right\}\right] d\theta_i = \bar{n}.$$

Substituting for the distribution of costs and solving for $\bar{n}$ yields:

$$\begin{aligned} \frac{\bar{n}}{\lambda} &= \left(\frac{\int_{-\alpha}^{\alpha} \max\left\{\Omega^A(\theta_i), \Omega^B(\theta_i)\right\} d\theta_i}{\alpha C \lambda}\right)^{1/2} = \sqrt{\frac{K}{C\lambda}} \\ \bar{n} &= \sqrt{K\,\lambda\,/\,C}, \end{aligned}$$

where $K = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} \max\left\{\Omega^A(\theta_i), \Omega^B(\theta_i)\right\} d\theta_i$ represents the *voters' average valuation of the expected effect of their vote.* ■

Appendix 3: Stylized facts

There is a huge amount of empirical research on abstention in the U.S. Here, I only stress some of its most salient findings (More detail can be found in Mueller (1989) and Aldrich (1993, 1997) for instance).

From a survey study on presidential elections, Riker and Ordeshook (1968) conclude that, in the formulation $P\,B + D - C$, where P is the pivot probability, B is the benefit of changing the outcome of the election, D is some fixed benefit of voting and C the gross cost of voting, all variables influence the probability that a voter turns out at the election:

Table 3: Riker and Ordeshook's main conclusions.

		% turnout		% turnout
Close margin:	yes:	78%	no:	72%
Benefit (B):	high:	82%	low:	66%
Sense of duty (D):	high:	87%	low:	51%

Ashenfelter and Kelley (1975) give more mixed evidence: closeness does not seem to affect turnout (correct sign, but insignificant), which suggests that instrumental motivations would only weakly affect voting behaviour. By contrast, benefits and costs matter strongly (a $6-voting tax was abolished in 1972. The probability for a given individual to turn out was 42% lower with the tax), as well as the sense of duty (people who feel "obliged to" turn out with a probability increased by 30%). There is however controversy about how one can delineate B from D with the available data.

Concerning the variable P, Foster (1984) used data from presidential elections to reestimate –in cross-section– some previous time-series analyses. His conclusion is that the closeness of the election and the size of the electorate (which decreases P) are in general insignificant or of the wrong sign.

To summarize: "*Many applications, especially those that use aggregate data find that the P term is a significant predictor (e.g. Barzel and Silberberg, 1973; Settle and Abrams, 1976; Silberman and Durden, 1975). Other tests using survey data (e.g., Ferejohn and Fiorina, 1975; Foster, 1984) have found it to be unrelated to the vote*" (Aldrich, 1993). But, in cross-section, any variation in perceived closeness must come from personal errors in answering P-like questions, since all different answers regard the same event, while time-series studies compare true differences in closeness (Aldrich, 1976, 1993). Finally, there is a clear upward trend in abstention rates, and turnout is lower in legislative than in presidential elections.

Another set of studies has underlined the **class bias** of turnout: richer and more educated voters tend to participate more than others. This observation is made in presidential as well as in legislative elections. There is less clear evidence about the way this class bias evolves over time: for some, it is stable. For others, it is falling, even if there is more evidence about the latter (e.g. see Wolfinger and Rosenstone, 1980; Cavanagh, 1982; Teixeira, 1987; Leighley and Nagler, 1992; Shields and Goidel, 1997).

This gives us four stylized facts and one point of debate:

Stylized Fact 1: Sense of duty motivates participation, costs of voting reduces participation.

Stylized Fact 2: There is a negative trend in turnout since the end of World War II.

Stylized Fact 3: Participation increases in education and income.

Stylized Fact 4: Participation varies with the type of election.

Point of debate: Does closeness of elections increase turnout?

The third stylized fact is considered as counterintuitive. Education and income "pick up the wrong sign" in regressions. Indeed, more educated people should be aware that pivot probabilities are negligible. Richer people should also have higher opportunity cost of time, leading to higher costs of voting and consequently to lower turnout rates. Mueller (1989) argues observations can be explained by the fact that successful people also learn to comply to social rules, which may explain their behavior. However, if the outcome of the election matters more for richer people, the model would predict higher turnout rates for these people, in which case the sign of these regressors are not "wrong".

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FIGURES

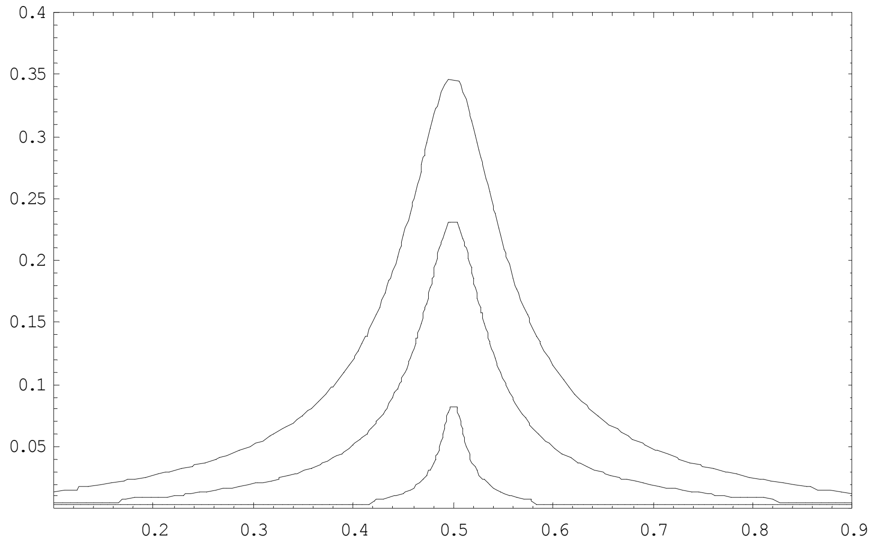

Figure 1. Expected turnout rate t (vertical axis) in function of γ_A (horizontal axis) for $\lambda = 1\,500$ (upper curve), $\lambda = 5\,000$ (intermediate curve) and $\lambda = 100\,000$ (bottom curve).

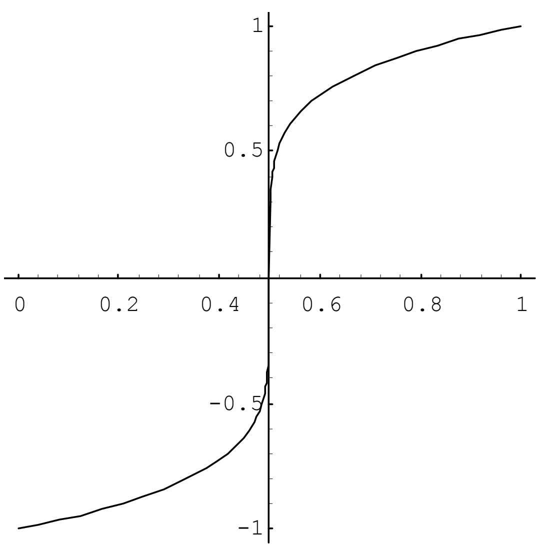

Figure 2. Implemented platform if $A = -1$ and $B = 1$. Horizontal axis: share of B.

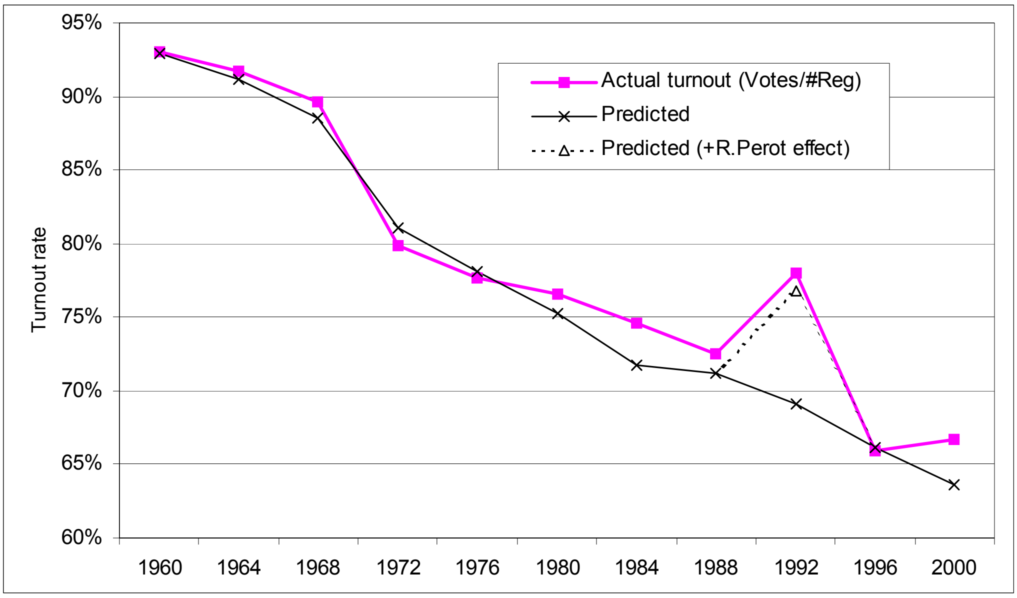

Figure 3. Calibrating the model on U.S. data.